Transition Dynamics in the Aiyagari Model, with an Application to Wealth Tax

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Abstract

I first replicate the steady-state of the Aiyagari (1994) model, and then analyze the model's transition dynamics after the government unexpectedly imposes a (permanent) wealth tax. I also compute the welfare costs of this policy change for each consumer and the aggregate economy. The codes can be found at https://sites.google.com/site/toshimukoyama/lecture-notes-and-slides. The direct link to the codes is https://sites.google.com/site/toshimukoyama/Aiyagari_codes.zip.

1 Introduction

The primary purpose of this note is to (i) explain the computation (replication) of Aiyagari (1994) model; (ii) explain the computation of the deterministic transition dynamics of the Aiyagari (1994) model after an unexpected policy change; and (iii) use the model to analyze the effects of wealth tax.

Here I mainly highlight the benefit of analyzing the transition dynamics in the long-run policy analysis. Often economists compare the individual welfare across steady-states to make normative assessments of a policy. In an economy with infinitely-lived agents, it is difficult to justify this exercise because it often requires resources to move from one steady-state to another.[1] The welfare result can even be reversed when the transition dynamics is taken into account.[2] In a life-cycle economy, the steady-state comparison can occasionally be justified if the welfare effects on future newborns are analyzed because these newborns do not bear the burden of transition. Even in a life-cycle economy, however, an explicit analysis of the transition dynamics is often relevant, because, for example, the welfare impact on the current generation is an important factor in analyzing the political decision-making.

While the effects of unanticipated policy changes (shocks) have been analyzed since the 1980s, the properties of this particular type of policy experiment have not been understood until recently. For example, Mukoyama (2010) shows that these policy changes can involve wealth transfers across agents, even when the asset markets are complete. The transfers occur because the contingency claims on these policy changes cannot be traded ex ante, as these events are not foreseen. Recently, these shocks are named "MIT shocks" and have been utilized in wider contexts. For example, Boppart et al. (2018) utilizes the transition dynamics from this type of shocks in analyzing heterogeneous-agent models with aggregate shocks.[3]

2 Model

The model is a discrete-time with an infinite horizon. Consumers supply an exogenous amount of labor and receive wage income, and accumulate assets in the form of

[1]In the context of the neoclassical growth model, this is the contrast between the Golden Rule and the Modified Golden Rule.

[2]Mukoyama (2013) makes this point in the context of the unemployment insurance policy. See the working paper version for further analysis: `https://sites.google.com/site/toshimukoyama/UIpolicyOld.pdf`.

[3]See, Krusell et al. (2020), for an application of this computational method. The codes can be found at `https://sites.google.com/site/toshimukoyama/KMRS_replication.zip`.

physical capital and rent out capital to firms. Firms produce the final goods (used for consumption and investment by consumers) using capital and labor. Both firms and consumers act competitively.

The labor supply is stochastic, and this shock is idiosyncratic. The most important assumption is that the insurance (contingency claim) market for this idiosyncratic shock is missing. The consumer can self-insure by accumulating the asset (capital), subject to a borrowing constraint.

2.1 Consumers

Consumers live forever. The utility function is assumed to be

$$E_0\left[\sum_{t=0}^{\infty}\beta^t U(c_t)\right],$$

where $\beta \in (0,1)$ is a discount factor, c_t is consumption at period t, and $U(\cdot)$ is an increasing and concave utility function. The budget constraint for the consumer is

$$c_t + a_{t+1} = w_t\ell_t + (1 + r_t - \delta - \tau_t)a_t + T_t,$$

where a_t is the asset (capital) holding at period t, w_t is the wage rate, ℓ_t is the labor supply, r_t is the rental rate of capital, and δ is the depreciation rate. The exogenous labor supply ℓ_t is random and follows a Markov process. What is different from Aiyagari (1994) here is the inclusion of the wealth tax τ_t and the transfer T_t. The tax rate τ_t is set as a policy variable and the government transfers the tax income to consumers as a lump-sum transfer T_t.[4] I set $\tau_t = 0$ at the initial steady state, and consider an unexpected and permanent switch to $\tau_t = 0.005$. The economy will experience the transition dynamics in which w_t and r_t changes over time, and eventually the economy settles in the new steady state. The consumer also faces the borrowing constraint:

$$a_{t+1} \geq b,$$

where $b \leq 0$.

The consumer's Bellman equation in a steady state is

$$V(a,\ell) = \max_{c,a'} U(c) + \beta E[V(a',\ell')|\ell] \tag{1}$$

[4]The implicit timing assumption within period is that first the production occurs using capital and labor, capital depreciates, the rental rate and the wage are paid. Then the government collects the tax based on the beginning-of-the period value of a, and the transfer happens. Finally the consumption-saving decision is made.

subject to

$$c + a' = w\ell + (1 + r - \delta - \tau)a + T$$

and

$$a' \geq b.$$

After the policy change, the Bellman equation have to be modified so that w_t, r_t, and T_t are time-varying (but the consumer has perfect foresight on all future values) and thus the value function also depends on the time since the policy change.

2.2 Firms

The production function for the firms is

$$Y_t = K_t^{\alpha} L_t^{1-\alpha}.$$

2.3 Government

The government repays all the wealth tax that it receives. Thus the government budget constraint is

$$T_t = \tau \int a_t(i) di,$$

where i is the index of consumers.

2.4 Equilibrium

The equilibrium conditions of the capital and labor markets are

$$K_t = \int a_t(i) di$$

and

$$L_t = \int \ell_t(i) di.$$

Note that L_t is exogenous and, from the law of large numbers, is constant over time.

From the firm's first-order conditions,

$$r_t = \alpha \left(\frac{K}{L} \right)^{\alpha - 1} \tag{2}$$

and

$$w_t = (1 - \alpha) \left(\frac{K}{L} \right)^{\alpha}. \tag{3}$$

3 Calibration

Calibration mostly follows Aiyagari (1994). Aiyagari considers various combinations of the parameter values but I pick one of them here. The utility function is assumed to be $U(c) = \log(c)$. Discount factor $\beta = 0.96$, the capital share $\alpha = 0.36$, and the depreciation rate $\delta = 0.08$. The borrowing constraint is set at $b = 0$.

The stochastic process for ℓ_t is assumed to be

$$\log(\ell_t) = \rho \log(\ell_{t-1}) + \sigma(1 - \rho^2)^{1/2}\epsilon_t,$$

where ϵ_t follows $N(0, 1)$. We set $\rho = 0.9$ and $\sigma = 0.4$. The AR(1) process is approximated by a Markov chain using Tauchen's (1986) method.

4 Computation

The codes can be found at my website: `https://sites.google.com/site/toshimukoyama/lecture-notes-and-slides`. The direct link to the codes is `https://sites.google.com/site/toshimukoyama/Aiyagari_codes.zip`. The zip file also contains a short readme file. The computation is structured as follows.

1. First, compute the initial steady-state and the final steady-state. The steps are identical.

   (a) After initializing parameters, guess K/L and T (in the case of $\tau > 0$). With (2) and (3), we can solve for the Bellman equation (1). I use the value-function iteration with a cubic interpolation.[5] I put more grids for smaller values of a. I use the Golden Section Search for optimization. (Given that the problem is smooth, I could use an alternative method but this is sufficiently fast.)

   (b) Compute the invariant distribution of (a, ℓ). Aiyagari (1994) uses a simulation-based method but I use the iteration over density function, as described in Heer and Maußner (2005) and Young (2010). I put equally-spaced grid over a.[6]

[5] The results are very similar if I use a linear interpolation.

[6] In my earlier lecture note on Krusell and Smith (1998) model (Mukoyama (2019); can be found at `https://sites.google.com/site/toshimukoyama/lecture-notes-and-slides`, codes at `https://sites.google.com/site/toshimukoyama/KrusellSmithlog.zip`), I recommended log-spaced grids for this purpose. The reason for this difference is that, in that model, the asset distribution can potentially have a very long (and fat) tail. I could still use the log-spaced grids here, but the marginal benefit from doing so is less.

(c) Compute K/L and T from the invariant distribution and compare with the guess. Iterate until the guess is correct.

2. For computing the transition, first import the value functions and decision rules (for both the initial state and the terminal state) from the steady-state calculations above. Also import the invariant distribution at the initial state. Solve the value function and the decision rule once again, before using, to improve accuracy.

3. Guess the time series of K_t/L_t and T_t for $t = 1, ...\mathbf{T}$, where $t = 1$ is the period when the unexpected change in τ happened and $t = \mathbf{T}$ is sufficiently far in future so that I can safely assume that by time $\mathbf{T}$ the economy is sufficiently close to the new steady state. Now I can obtain r_t and w_t (2) and (3). Also note that now $\tau = 0.005$ for all t. Assume that at $t = \mathbf{T}+1$, the economy is in the steady state with $\tau = 0.005$. Then I can use the terminal value function from the previous step in the right-hand side of the period-$\mathbf{T}$ Bellman equation (the nonstationary version of (1)). From there, with backward induction, the value functions and the decision rules for $t = 1, ..., \mathbf{T}$ can be solved.

4. Using the decision rule above, the economy can be simulated (again, using the density function) forward, starting from the uploaded density function at the initial steady state. Then K_t/L_t and T_t for $t = 1, ...\mathbf{T}$ can be computed. Compare this with the guess. Modify and iterate until convergence.

5. One can compute the welfare cost of the wealth tax by computing the $\lambda(i)$ that satisfies

$$E_1\left[\sum_{t=1}^{\infty}\beta^t U((1+\lambda(i))c_t(i)^{\tau=0})\right] = E_1\left[\sum_{t=1}^{\infty}\beta^t U(c_t(i)^{\tau=0.005})\right],$$

where $c_t(i)^{\tau=0}$ is consumer i's path of consumption under $\tau = 0$ (no policy change) and $c_t(i)^{\tau=0.005}$ is her path of consumption (including the transition path of the aggregate variables) under policy change. If $\lambda(i) > 0$, the consumer i is better off with the positive wealth tax and $\lambda(i) < 0$ implies that the consumer is worse off. With log utility,

$$\lambda(i) = \exp[V^{\tau=0.005}(a(i), \ell(i)) - V_1^{\tau=0}(a(i), \ell(i))] - 1,$$

where $V_1^{\tau=0.005}(a(i), \ell(i))$ is the value function computed above for period $t = 1$ and $V_1^{\tau=0}(a(i), \ell(i))$ is the value function from the initial steady state.

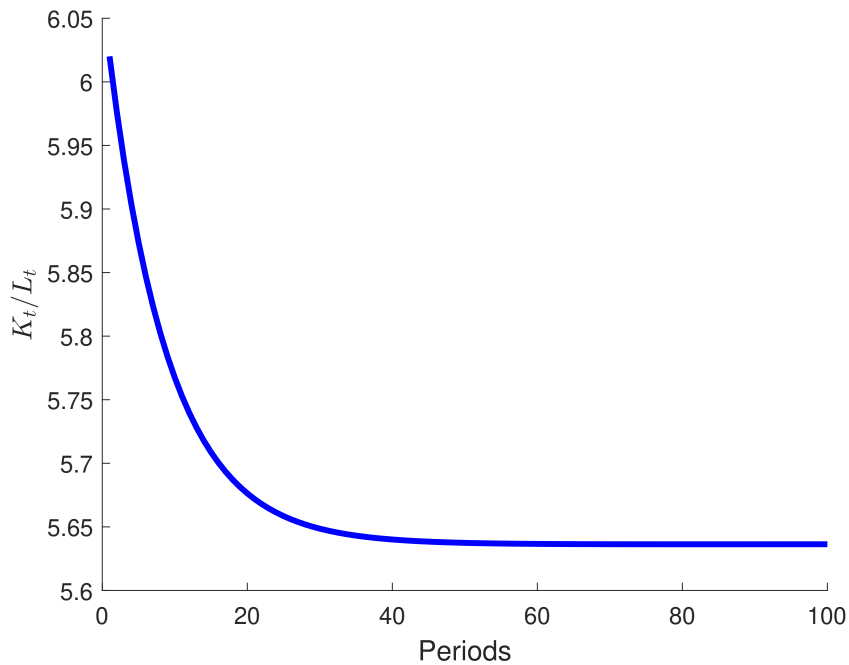

Figure 1: The time series of K_t/L_t

5 Results

The result in the initial steady-state replicates Aiyagari (1994). The value of $r - \delta$ is 3.41% and the saving rate is 25.2%, while in Aiyagari (1994) the corresponding numbers (from Table II) are 3.31% and 25.5%. (The results match well also with other combinations of ρ and σ.) I suspect that the somewhat minor differences come from the fact that Aiyagari (1994) uses Monte-Carlo simulations for computing the invariant distribution.

Figure 1 plots the time series of K_t/L_t. Here L_t is exogenous and constant, so this path reflects the movement of K_t. The reduction of K_t reflects two effects: (i) reduction of asset accumulation due to the decline in the net return $(r_t - \delta - \tau)$ and (ii) reduction in precautionary saving motive, because a larger T_t can act as an additional insurance.

Figure 2 draws the values of $\lambda(i)$ for individual i for different values of initial $a(i)$ and $\ell(i)$. For ℓ dimension, not to clutter the figure, I have drawn three curves (instead of seven) for each value of the initial ℓ. For example, "ℓ grid 1" means the first (the smallest) value of ℓ at the time of the policy change. The figure also describes the 25th-percentile, median, and the 75th-percentile of the overall wealth distribution.

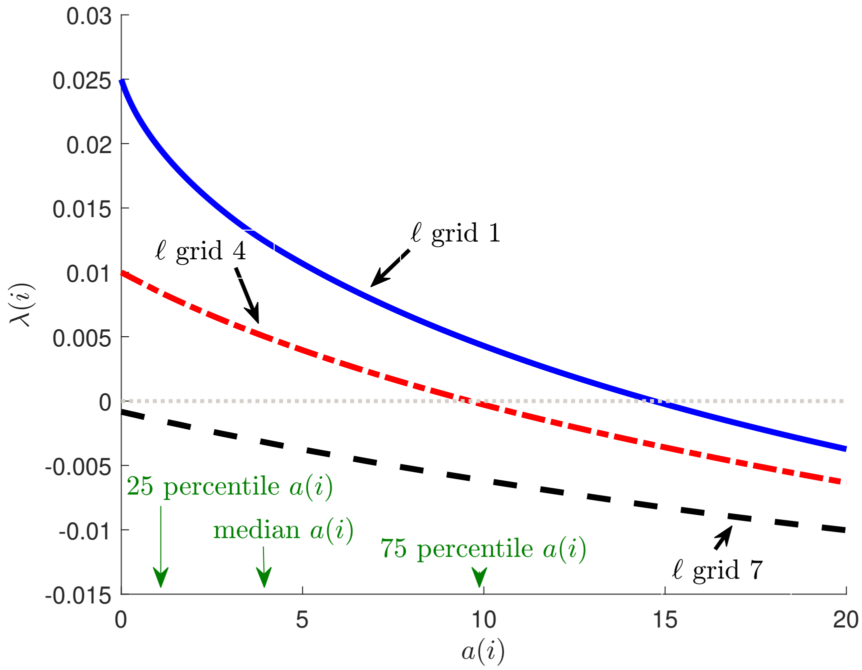

Figure 2: The values of $\lambda(i)$ for different values of a and ℓ at the initial period

One can see from the figure that there is substantial heterogeneity in the welfare effects. Given that the wealth tax involves a transfer from an asset-rich to asset-poor, it is not surprising that $\lambda(i)$ is decreasing in $a(i)$. It is also decreasing in ℓ, because a high ℓ today indicates that the agent will become an asset-rich in future. There are two additional effects. First, because T_i can act as an insurance for the idiosyncratic shock, an increase in wealth tax (and therefore an increase in transfer) can benefit especially the wealth-poor as insurance. Second, the change in K/L induces changes in r and w. In particular, a decrease in K/L increases in r and decrease w, benefiting an agent with high a and hurting an agent with high ℓ. Although the after-tax net return $r - \tau$ decreases as τ increases, the increase in τ is almost offset by the increase in r. The value of $r - \tau$ in the initial steady state is 3.412% whereas it is 3.410% in the terminal steady-state.[7]

The average value of λ, which is defined as

$$\bar{\lambda} \equiv \int \lambda(i) di,$$

[7]The welfare decomposition into various factors can be conducted more formally. See Mukoyama (2013) and its working paper version https://sites.google.com/site/toshimukoyama/UIpolicyOld.pdf.

is 0.41%.[8] Lining up the consumers in the order of the loss from the wealth tax, a median consumer gains 0.42% from the introduction of wealth tax. Therefore, this particular wealth tax reform would be supported in a majority-voting setup. In fact, about 70% of the consumers experience $\lambda(i) > 0$.

[8]See Mukoyama (2010) for a discussion on different methods of aggregating $\lambda(i)$.

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