Constraints in the Consumption-Saving Problem

Toshihiko Mukoyama

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In this note, I will show some equivalence results for the constraints in the standard consumption-saving problem.

1 Finite-horizon problem

Let us start from a finite-horizon consumption-saving problem. The planning horizon is from 0 to T. The consumer maximizes her utility:

$$\max_{\{c_t, a_{t+1}\}_{t=0}^T} \sum_{t=0}^{T} \beta^t u(c_t),$$

subject to the budget constraint. The consumer earns $w_t \geq 0$ during each period t, and has an asset a_t at the beginning of time t. The asset provides ra_t interest income (where $r > 0$) during period t. The flow budget constraint for $t = 0, ..., T$ is

$$a_{t+1} = (1 + r)a_t + w_t - c_t. \tag{1}$$

It is straightforward to consolidate the flow budget constraint to obtain the intertemporal budget constraint:

$$\sum_{t=0}^{T} \frac{c_t}{(1+r)^t} + \frac{a_{T+1}}{(1+r)^T} = (1 + r)a_0 + \sum_{t=0}^{T} \frac{w_t}{(1+r)^t}. \tag{2}$$

In fact, (1) and (2) are equivalent in the sense that one can create (1) from (2) as well.

In both (1) and (2), the consumer is allowed to choose any value for a_{T+1}, she would want to (and will be able to) consume a lot by going very negative on a_{T+1} and the maximization problem won't be well-defined. Thus, one natural requirement in many environments would be to impose

$$a_{T+1} \geq 0 \tag{3}$$

in addition to (2) or (1). Using (3), (2) can be rewritten as

$$\sum_{t=0}^{T} \frac{c_t}{(1+r)^t} \leq (1 + r)a_0 + \sum_{t=0}^{T} \frac{w_t}{(1+r)^t}. \tag{4}$$

The constraint (4) is the standard lifetime budget constraint.

Now, let us ask the following question: with this budget constraint, how far can we allow this consumer to borrow at each point in time $t = 0, ..., T-1$ (by construction, there is no borrowing at T if $a_{T+1} \geq 0$ is imposed)? In other words, how low can a_{t+1} go, without allowing for bankruptcy, at $t = 0, ..., T-1$? Clearly, when a_{t+1} becomes very low, it is impossible to satisfy $a_{T+1} \geq 0$ even without consuming at all (and using all income for repayment) in subsequent periods. One can impose a no-borrowing constraint $a_{t+1} \geq 0$ for all t, but this restriction is too strong: if $w_s > 0$ for some $s \geq t+1$, a very small amount of borrowing at time t can be repaid by using the subsequent w_s. In fact, the maximum that the consumer can repay is exactly the amount that can be repaid with zero consumption in subsequent periods. Using (1) from time $t+1$ to T,

$$\frac{a_{T+1}}{(1+r)^T} = \frac{a_{t+1}}{(1+r)^t} + \sum_{s=t+1}^{T} \frac{w_s - c_s}{(1+r)^s} \tag{5}$$

holds. Thus, $a_{T+1} \geq 0$ implies

$$\frac{a_{t+1}}{(1+r)^t} + \sum_{s=t+1}^{T} \frac{w_s - c_s}{(1+r)^s} \geq 0.$$

Therefore, setting $c_s = 0$ for $s = t+1, ..., T$,

$$a_{t+1} \geq -\sum_{s=t+1}^{T} \frac{w_s}{(1+r)^{s-t}} \tag{6}$$

is the constraint that has to be imposed for $t = 0, ..., T-1$, in order to ensure repayment. In other words, if a_{t+1} becomes below the right-hand side value of (6), it is impossible for the consumer to repay and satisfy $a_{T+1} \geq 0$.

In sum, I have shown that the following three are equivalent:

(i) The flow budget constraint (1) for $t = 0, ..., T$ and the terminal condition (3)

(ii) The flow budget constraint (1) for $t = 0, ..., T$, the borrowing limit (6) for $t = 0...., T-1$ and the terminal condition (3)

(iii) The lifetime budget constraint (4)

Of course **(ii)** is redundant because (1) and (3) would imply (6), but it is useful to know how much one can borrow at each period.

2 Infinite horizon

Next, consider the case of infinite horizon. The consumer maximizes his utility:

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t),$$

subject to the budget constraint.

In this case, the final period doesn't exist, so one might think that a constraint that is similar to (3) is not necessary anymore. However, if no constraint is imposed in addition to (1), it is possible to borrow $b > 0$ at time t, repay $(1+r)b$ by borrowing $(1+r)b$ at time $t+1$, repay $(1+r)^2 b$ by borrowing $(1+r)^2 b$ at time $t+2$ and so on, and keep rolling the debt, without ever repaying. This type of scheme is called a "Ponzi game." Allowing Ponzi-type schemes makes the maximization problem not well defined, because b can be an any number and therefore a rational consumer wants to borrow a lot and keep borrowing. Also, the Ponzi-scheme cannot be sustained forever in reality, since at some point the interest payment $r(1+r)^T b$ will exceed all resources in the universe. Thus we would actually want to impose some additional constraint that is (in effect) similar to (3) in the finite horizon case.

One might think that simply taking a limit $\lim_{T\to\infty} a_{T+1} \geq 0$ would suffice, but unfortunately it is not the case. Suppose, for example, $w_t = \bar{w} > 0$ for all t. Then, there is nothing wrong about allowing the consumer to borrow a small amount at any period, for example $\varepsilon < \bar{w}/(1+r)$, since the consumer can repay this borrowing by his income in the following period. (Note that this is different from the Ponzi scheme since he is repaying by his own income, rather than financing the repayment by borrowing.) But then, the outcome of this scheme $a_{t+1} = -\varepsilon$ for all t, does not satisfy $\lim_{T\to\infty} a_{T+1} \geq 0$. Therefore $\lim_{T\to\infty} a_{T+1} \geq 0$ is "too strong" as a constraint. (We will go back to this example in footnote 1.) The question is: what is the "maximally loose" but yet reasonable constraint?

Let us start by assuming the flow budget constraint (1) always has to hold. In the finite horizon case, (2) holds up to time T by combining (1). Thus it seems reasonable to take the limit in (2) to $T \to \infty$ and impose

$$\lim_{T\to\infty} \frac{a_{T+1}}{(1+r)^T} \geq 0 \tag{7}$$

in order to ensure that

$$\sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t} \leq (1+r)a_0 + \sum_{t=0}^{\infty} \frac{w_t}{(1+r)^t} \tag{8}$$

is the intertemporal (lifetime) budget constraint. The condition (7) is called "no Ponzi game" condition, since it (in effect) prohibits a Ponzi-type scheme. (Of course, for (8) to make sense, we have to assume that $\sum_{t=0}^{T} w_t/(1+r)^t$ remains finite as $T \to \infty$.)

An alternative is to impose a borrowing constraint that is similar to (6). In the infinite horizon,

$$a_{t+1} \geq -\sum_{s=t+1}^{\infty} \frac{w_s}{(1+r)^{s-t}}. \tag{9}$$

This borrowing limit is called the "natural borrowing limit."[1] In fact, we can show that imposing (9) for all t is equivalent to imposing the no Ponzi game condition (7).

Below I will formally show that the following three are equivalent: (1).

[1] In the special case of $w_t = \bar{w}$ for all t, this can be rewritten as $a_{t+1} \geq -\bar{w}/r$. With the similar argument

(i) The flow budget constraint (1) for $t = 0, 1, ...$ and no Ponzi game condition (7)

(ii) The flow budget constraint (1) for $t = 0, 1, ...$ and the natural borrowing limit (9) for $t = 0, 1, ...$

(iii) The lifetime budget constraint (8)

In order to show the equivalence, we will start from showing that (i)⇒(ii), then (ii)⇒(iii), and then (iii)⇒(i).

(i)⇒(ii): [2] From (1), (5) also holds. Rewriting (5):

$$a_{t+1} = (1+r)^t \frac{a_{T+1}}{(1+r)^T} - \sum_{s=t+1}^{T} \frac{w_s - c_s}{(1+r)^{s-t}}.$$

Taking $T \to \infty$,

$$a_{t+1} = (1+r)^t \lim_{T\to\infty} \frac{a_{T+1}}{(1+r)^T} - \sum_{s=t+1}^{\infty} \frac{w_s - c_s}{(1+r)^{s-t}}.$$

Using (7) this equation implies

$$a_{t+1} \geq - \sum_{s=t+1}^{\infty} \frac{w_s - c_s}{(1+r)^{s-t}}.$$

Because $c_s \geq 0$ for all s,

$$a_{t+1} \geq - \sum_{s=t+1}^{\infty} \frac{w_s}{(1+r)^{s-t}}$$

holds, which is (9). Since t was arbitrary, we are done.

(ii)⇒(iii): To show that (9) for all t implies (8), first note (9) implies

$$\frac{a_{t+1}}{(1+r)^t} \geq - \sum_{s=t+1}^{\infty} \frac{w_s}{(1+r)^s}. \tag{10}$$

The flow budget constraint (1) implies (2) holds for any T:

$$\frac{a_{T+1}}{(1+r)^T} = (1+r)a_0 + \sum_{t=0}^{T} \frac{w_t - c_t}{(1+r)^t}.$$

as in the finite case, it is sufficient to impose $\lim_{T\to\infty} a_{T+1} \geq -\bar{w}/r$, rather than imposing $a_{t+1} \geq -\bar{w}/r$ every period, because once $a_{t+1} \geq -\bar{w}/r$ is violated at some period t, it becomes impossible to satisfy $\lim_{T\to\infty} a_{T+1} \geq -\bar{w}/r$.

[2] In an earlier note, I had a different proof here. This proof was suggested by Latchezar Popov.

Thus, combining with (10),

$$(1+r)a_0 + \sum_{t=0}^{T} \frac{w_t - c_t}{(1+r)^t} \geq - \sum_{t=T+1}^{\infty} \frac{w_t}{(1+r)^t}.$$

holds. Rearranging and taking $T \to \infty$ (the right-hand side converges to zero[3]) results in (8).

(iii)⇒(i): Note that because (8) doesn't specify $a_1, a_2, ...$, one can create (2) by appropriately defining a_T. In turn, one can also create (1) that corresponds these $a_1, a_2, ...$, because (2) and (1) are equivalent.

Take a limit of (2) for $T \to \infty$,

$$\sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t} + \lim_{T\to\infty} \frac{a_{T+1}}{(1+r)^T} = (1+r)a_0 + \sum_{t=0}^{\infty} \frac{w_t}{(1+r)^t}.$$

Rewriting,

$$\lim_{T\to\infty} \frac{a_{T+1}}{(1+r)^T} = (1+r)a_0 + \sum_{t=0}^{\infty} \frac{w_t}{(1+r)^t} - \sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t}.$$

From (8), this equation implies (7).

■

In sum, there are two equivalent ways to ensure that the flow budget constraint (1) imply the intertemporal budget constraint (8) in the infinite horizon case. The first is to impose the no Ponzi game condition (7) in addition to (1), and the second is to impose the natural borrowing limit (9) on top of (1).[4]

[3] This result follows from $\sum_{t=0}^{\infty} \frac{w_t}{(1+r)^t}$ being finite. The right-hand side can be rewritten as

$$-\sum_{t=0}^{\infty} \frac{w_t}{(1+r)^t} + \sum_{t=0}^{T} \frac{w_t}{(1+r)^t},$$

whose limit with $T \to \infty$ equals zero.

[4] Similarly to the finite-horizon case, there is a sense of "redundancy" for the natural borrowing limit. If (9) is satisfied at time t, it is enough to ensure that it is satisfied for all $s < t$. (If it is violated at time $s < t$, it is impossible to satisfy it for time t.) Thus only the "far future" natural borrowing limit is relevant. But if we take a limit of (10) (which is a simple transformation of (9)) for $t \to \infty$, we obtain (7), which is an analogue of the terminal condition (3).