DMP Model in Continuous Time

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1 Unemployment dynamics and Beveridge curve

We start from a discrete-time formulation. Let the period length be Δ. The economy has a continuum of population 1 workers who are employed or unemployed. The unemployment dynamics can be represented as

$$u(t+\Delta) = \sigma\Delta(1-u(t)) + (1-\lambda_w\Delta)u(t),$$

where $u(t)$ is the unemployment rate (which is equal to the unemployment population, because the total population is one) at time t, $\sigma\Delta > 0$ is the probability of losing the job and becoming unemployed over the period Δ, $\lambda_w\Delta > 0$ is the probability of finding a job over the period Δ. The first term on the right-hand side is the employed worker moving into unemployment, and the second term is the unemployed worker staying unemployed.

One can rewrite this equation as

$$u(t+\Delta) - u(t) = \sigma\Delta(1-u(t)) + \lambda_w\Delta u(t),$$

This equation relates the change in stock (net flow) on the left-hand side to the gross worker flows on the right-hand side. The first term on the right-hand side is the inflow into the unemployment pool, and the second term is the outflow from the unemployment pool.

Dividing both sides by Δ, we obtain

$$\frac{u(t+\Delta) - u(t)}{\Delta} = \sigma(1-u(t)) + \lambda_w u(t),$$

and taking $\Delta \to 0$, the continuous-time dynamics is

$$\dot{u}(t) = \sigma(1-u(t)) + \lambda_w u(t), \tag{1}$$

where $\dot{u}(t) = du(t)/dt$ is the time derivative.

The first step of the baseline DMP model is to endogenize the job-finding rate λ_w. For that purpose, we introduce a function called the matching function. The matching function takes the aggregate vacancy v and the aggregate unemployment rate u as inputs and the number of matches as the output. The matching function takes the form of

$$M(u,v)\Delta.$$

We assume that the matching function is increasing in both terms and exhibits constant returns to scale. We also assume that matching is random; that is, all vacancies have an equal chance to match with an unemployed worker, and all unemployed workers have an equal chance to match with a vacancy.

With these assumptions, the probability of a vacancy matching with an unemployed worker during the period Δ is

$$\frac{M(u,v)\Delta}{v} = M\left(\frac{1}{\theta}, 1\right)\Delta = \lambda_f(\theta)\Delta,$$

where

$$\theta \equiv \frac{v}{u}$$

is the labor market tightness, and the probability of an unemployed worker matching with a vacancy during the period Δ is

$$\frac{M(u,v)\Delta}{u} = M(1,\theta)\Delta = \lambda_w(\theta)\Delta = \theta\lambda_f(\theta)\Delta.$$

Note that, with our assumptions on $M(u,v)$, $\lambda_f(\theta)$ is decreasing in θ and $\lambda_w(\theta)$ is increasing in θ.

With this formulation, (1) becomes

$$\dot{u}(t) = \sigma(1-u(t)) + \lambda_w(\theta(t))u(t), \tag{2}$$

where $\theta(t) = v(t)/u(t)$ when $v(t)$ may change over time. Consider the steady state where $v(t)$ is constant at v and $\dot{u}(t) = 0$. Then, from (2), the steady-state unemployment rate $\bar{u}$ satisfies

$$0 = \sigma(1-\bar{u}) + \lambda_w(v/\bar{u})\bar{u}$$

or

$$\bar{u} = \frac{\sigma}{\lambda_w(v/\bar{u}) + \sigma}. \tag{3}$$

Alternatively, we can use the matching function and rewrite

$$M(\bar{u}, v) + \sigma\bar{u} = \sigma.$$

The left-hand side is increasing in both $\bar{u}$ and v, and therefore, this equation represents a negative relationship between v and $\bar{u}$. This relationship (3) will be referred to as the *Beveridge curve condition*, as it corresponds to the Beveridge curve observed in the data.

2 Diamond-Mortensen-Pissarides model

2.1 Workers and firms

The Diamond-Mortensen-Pissarides (DMP) builds on the unemployment dynamics in the previous section and adds the endogenous determination of the aggregate vacancy $v(t)$. Firms create vacancies by trading off the current vacancy-posting cost and future profit from the firm-worker match. We assume that each firm (vacancy) hires one worker. To analyze the value of creating a vacancy, we compute the following present values.

We assume that workers do not borrow or save, and consume the current income. The workers have linear utility and discount rate r. Let $w(t)$ be the equilibrium wage (which we will determine later). The present expected value of being employed, in the discrete-time setting with period length Δ, is

$$W(t) = w(t)\Delta + \frac{1}{1+r\Delta}((1-\sigma\Delta)W(t+\Delta) + \sigma\Delta U(t+\Delta)), \tag{4}$$

where $U(t)$ is the value of being unemployed, expressed as

$$U(t) = b\Delta + \frac{1}{1+r\Delta}(\lambda_w(\theta(t))\Delta W(t+\Delta) + (1-\lambda_w(\theta(t))\Delta)U(t+\delta)).$$

Multiplying $(1+r\Delta)$ on both sides of (4), we obtain

$$(1+r\Delta)W(t) = (1+r\Delta)w(t)\Delta + (1-\sigma\Delta)W(t+\Delta) + \sigma\Delta U(t+\Delta).$$

Moving $W(t)$ to the right-hand side and dividing by Δ,

$$rW(t) = (1+r\Delta)w(t) - \sigma(W(t+\Delta) - U(t+\Delta)) + \frac{W(t+\Delta)-W(t)}{\Delta}$$

holds, and taking $\Delta \to 0$ yields

$$rW(t) = w(t) - \sigma(W(t) - U(t)) + \dot{W}(t). \tag{5}$$

This equation is the continuous-time version of the Bellman equation (Hamilton-Jacobi-Bellman equation, HJB equation) for the value of employment. Similarly,

$$rU(t) = b + \lambda_w(\theta(t))(W(t) - U(t)) + \dot{U}(t) \tag{6}$$

is the HJB equation for unemployed workers.

Because of the one-firm to one-worker assumption, the firm is either matched with a worker or vacant. The value of a matched firm is

$$J(t) = p(t)\Delta - w(t)\Delta + \frac{1}{1+r\Delta}(\sigma\Delta V(t+\Delta) + (1-\sigma\Delta)J(t+\Delta)),$$

where $p(t)\Delta$ is the product produced by the match and $V(t)$ is the value of vacancy, computed by

$$V(t) = -\kappa\Delta + \frac{1}{1+r\Delta}(\lambda_f(\theta(t))\Delta J(t+\Delta) + (1-\lambda_f(\theta(t))\Delta)V(t+\Delta)),$$

where $\kappa\Delta$ is the cost of posting a vacancy. With similar procedures as above, the continuous-time HJB equations are

$$rJ(t) = p(t) - w(t) - \sigma(J(t) - V(t)) + \dot{J}(t) \tag{7}$$

and

$$rV(t) = -\kappa + \lambda_f(\theta(t))(J(t) - V(t)) + \dot{V}(t). \tag{8}$$

We assume that anyone can start a firm by posting a vacancy. Thus, if $V(t) > 0$, new entrants would keep posting vacancies until $V(t)$ becomes zero. Additional vacancies would drop the value of $V(t)$ because $\lambda_f(\theta(t))$ is a decreasing function of $\theta(t) = v(t)/u(t)$. Therefore, $V(t) = 0$ has to always hold, as long as $v(t) > 0$, which we assume is always the case.[1] We call the condition

$$V(t) = 0 \tag{9}$$

the *free-entry condition*.

2.2 Wage determination

In this framework, we cannot determine wages from the marginal product of labor. For example, if $p(t) > w(t)$, under perfect competition, another firm would poach the worker by offering a slightly higher wage, until $w(t)$ approaches $p(t)$. In this framework, for a worker to meet with another firm, the worker has to endure the cost of unemployment. Similarly, for the firm, losing a worker would imply waiting for another worker by incurring the vacancy cost. Thus, the matched worker and firm are in a bilateral monopoly situation. Each period, the match generates $p(t) - b$ amount of surplus (b is the opportunity cost for the worker). In the present value, the match has

$$(W(t) - U(t)) + (J(t) - V(t))$$

amount of surplus. In a bilateral monopoly situation, there is no competitive force to determine how to split the surplus. Here, we assume that the surplus is split by the level of wage that solves the following maximization problem:

$$\max_{w} \; (\tilde{W}(w,t) - U(t))^{\gamma}(\tilde{J}(w,t) - V(t))^{1-\gamma}, \tag{10}$$

where $\gamma \in (0,1)$. The functions $\tilde{W}(w,t)$ and $\tilde{J}(w,t)$ are the values of employed workers and a matched firm, but we made it explicit that only w in this period can move. That is, the wage is reset every period. The solution to this problem is often called *Generalized Nash Bargaining* rule. It is called "generalized" because the original Nash Bargaining maximizes the product (the "Nash product") of each party's surpluses (which is equivalent to $\gamma = 1/2$). Here, we allow different "weights" on the worker and the firm. Because a larger γ would result in a larger weight on the worker, γ is often referred to as the worker's bargaining power.

The functions $\tilde{W}(w,t)$ and $\tilde{J}(w,t)$ can formally be written as

$$\tilde{W}(w,t) = w\Delta + \frac{1}{1+r\Delta}((1-\sigma\Delta)W(t+\Delta) + \sigma\Delta U(t+\Delta)),$$

[1] It is possible that $V(t) < 0$ when $v(t) = 0$.

and

$$\tilde{J}(w,t) = p(t)\Delta - w\Delta + \frac{1}{1+r\Delta}(\sigma\Delta V(t+\Delta) + (1-\sigma\Delta)J(t+\Delta)).$$

The first-order condition for (10) is

$$\gamma\frac{\partial\tilde{W}(w,t)}{\partial w}(\tilde{W}(w,t)-U(t))^{\gamma-1}(\tilde{J}(w,t)-V(t))^{1-\gamma}+(1-\gamma)\frac{\partial\tilde{J}(w,t)}{\partial w}(\tilde{W}(w,t)-U(t))^{\gamma}(\tilde{J}(w,t)-V(t))^{-\gamma} = 0.$$

Because $\partial\tilde{W}(w,t)/\partial w = \Delta$ and $\partial\tilde{J}(w,t)/\partial w = -\Delta$, this equation can be rewritten as

$$(1-\gamma)(W(t) - U(t)) = \gamma(J(t) - V(t)), \tag{11}$$

where $W(t) = \tilde{W}(w^*, t)$ and $J(t) = \tilde{J}(w^*, t)$ are the equilibrium values (with the maximizer w^*).

2.3 Steady-state equilibrium

First, consider the steady-state equilibrium where $u(t)$ and $v(t)$ are constant. In equations (5), (6), (7), and (8), steady-state implies that $W(t)$, $U(t)$, $J(t)$, and $V(t)$ are constant over time. With the conditions (9) and (11), we will have six equations

$$rW = w - \sigma(W - U),$$

$$rU = b + \lambda_w(\bar{\theta})(W - U),$$

$$rJ = p - w - \sigma(J - V),$$

$$rV = -\kappa + \lambda_f(\bar{\theta})(J - V),$$

$$V = 0,$$

and

$$(1-\gamma)(W - U) = \gamma(J - V),$$

with six unknowns, W, U, J, V, $\bar{\theta}$ (steady-state value of θ), and w.

One can rearrange this set of equations to one equation with one unknown:

$$(1-\gamma)(p - b) - \frac{r + \sigma + \gamma\lambda_w(\bar{\theta})}{\lambda_f(\bar{\theta})}\kappa = 0. \tag{12}$$

This equation can be solved for $\bar{\theta}$. This condition is often referred to as the *job creation condition.*

2.4 Transition dynamics

Now, let us consider the transition dynamics by not imposing the steady-state conditions. Combining (5), (6), (7), (8), (9), and (11), we obtain

$$\dot{J}(t) = (1-\gamma)(p-b) - \frac{r+\sigma+\gamma\lambda_w(\theta(t))}{\lambda_f(\theta(t))}\kappa. \tag{13}$$

The right-hand side has the same expression as the equation (12).

From (8) and $\dot{V}(t) = 0$ (because $V(t)$ is always constant at zero), $J(t)$ can be expressed as

$$J(t) = \frac{\kappa}{\lambda_f(\theta(t))}.$$

Therefore,

$$\dot{J}(t) = -\frac{\kappa}{\lambda_f(\theta(t))^2}\lambda_f'(\theta(t))\dot{\theta}(t) = -\frac{\kappa}{\lambda_f(\theta)}\frac{\lambda_f'(\theta(t))\theta}{\lambda_f(\theta)}\frac{\dot{\theta}(t)}{\theta(t)} = J(t)\eta(\theta(t))\frac{\dot{\theta}(t)}{\theta(t)},$$

where $\eta(\theta(t)) \equiv -\lambda_f'(\theta(t))\theta/\lambda_f(\theta) > 0$ is the elasticity of matching function. Combining (12) and (13), together with this expression for $\dot{J}(t)$, one can see that $\dot{\theta}(t) > 0$ when $\theta(t) > \bar{\theta}$ and $\dot{\theta}(t) < 0$ when $\theta(t) < \bar{\theta}$. Thus, the only way that $\theta(t)$ does not diverge is for $\theta(t)$ to satisfy (13) with $\dot{J}(t) = 0$ all time. In other words, $\theta(t)$ has to satisfy the job creation condition even when the economy is not in a steady state.

Therefore, the transition dynamics of $u(t)$ and $v(t)$ are characterized by

$$\frac{v(t)}{u(t)} = \theta,$$

where θ satisfies the job creation condition (12):

$$(1-\gamma)(p-b) - \frac{r+\sigma+\gamma\lambda_w(\theta)}{\lambda_f(\theta)}\kappa = 0,$$

and the labor market dynamics (2)

$$\dot{u}(t) = \sigma(1-u(t)) + \lambda_w(\theta)u(t),$$

where θ is given above. Because $\dot{u}(t) > 0$ when $u(t) < \bar{u}$ and $\dot{u}(t) < 0$ when $u(t) > \bar{u}$, where $\bar{u} = \sigma/(\lambda_w(\theta)+\sigma)$ is the steady-state unemployment rate, the transition dynamics of $u(t)$ exhibits a monotone convergence to the steady-state value $\bar{u}$. Because $v(t)/u(t)$ is constant, $v(t)$ also exhibits the same dynamic property—it converges monotonically to $\bar{v} = \theta\bar{u}$.