Incomplete Market Models in Macroeconomics: Applications

Toshihiko Mukoyama
Georgetown University

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Today

- Today I will cover two applications of Aiyagari (1994)-type incomplete-market model.
  - Policy evaluations
  - Business cycle analysis

Last week's main references

- Mukoyama (2010). 'Welfare Effects of Unanticipated Policy Changes with Complete Asset Markets" *Economics Letters* 109, 104–178.
- Krusell, Mukoyama, and Smith (2011). "Asset Prices in a Huggett Economy," *Journal of Economic Theory* 146, 812-844.

Today's main references

- Mukoyama (2020). "Transition Dynamics in the Aiyagari Model, with an Application to Wealth Tax," Lecture notes, `https://toshimukoyama.github.io/MyWebsite/Aiyagari.pdf`
- Krusell, Mukoyama, Rogerson, and Şahin (2020). "Gross Worker Flows and Fluctuations in the Aggregate Labor Market," *Review of Economic Dynamics* 37, S205-S226. (Appendix talks about the computational method.)

Production economy

Aiyagari (1994).

- Production economy: competitive firms produce with the (aggregate) production function

$$Y_t = K_t^{\alpha} L_t^{1-\alpha}$$

  where Y_t is output, K_t is capital, and L_t is labor.

- Consumers: maximize

$$E_0 \left[\sum_{t=0}^{\infty} \beta^t U(c_t) \right]$$

  subject to

$$c_t + a_{t+1} = w_t \ell_t + (1 + r_t - \delta) a_t$$

  and

$$a_{t+1} \geq b.$$

  ℓ_t is random (and exogenous), a_t is capital stock holding.

Production economy

firm: $\max_{K,L} K^{\alpha}L^{1-\alpha} - rK - wL$

Aiyagari (1994).

- The prices w_t and r_t are determined in competitive markets:

$$w_t = (1-\alpha)\left(\frac{K_t}{L_t}\right)^{\alpha-1}$$

and

$$r_t = \alpha\left(\frac{K_t}{L_t}\right)^{\alpha}$$

where

$$K_t = \int a_t(i)di$$

and

$$L_t = \int \ell_t(i)di.$$

Supply

Production economy

Aiyagari (1994).

- This model looks very much like the neoclassical growth model (Ramsey), except that each individual behavior is consistent with the permanent-income (or life-cycle) hypothesis.
- This model (and its variations) has been used extensively in the policy analysis (especially fiscal policy): taxation, government debt, etc.
- Now, for these purposes, more people use finite-life (overlapping generations) versions of this model, because life-cycle elements are important for many policy questions (e.g. social security).
- Labor market policies can be analyzed, too, especially after ℓ_t process is endogenized (see, for example, Krusell et al., 2010).

Computation of Aiyagari (1994) in the steady state

- Recall that both w and r are functions of K/L. And both K and L can be computed by summing up individual a and ℓ in the stationary distribution.
- Individual optimization

$$V(a,\ell) = \max_{c,a'} \; U(c) + \beta E[V(a',\ell')|\ell]$$

subject to

$$c + a' = w\ell + (1 + r - \delta)a$$

and

$$a' \geq b$$

can be performed once w and r are known.
- Therefore, the steady state equilibrium can be computed with the following steps.

Computation of Aiyagari (1994) in the steady state

1. Guess K/L. (Because ℓ process is exogenous, L can independently computed by computing the stationary distribution of ℓ.) Compute w and r.
2. Perform the consumer's optimization.
3. Using the decision rule $a'(a, \ell)$ and the transition probabilities for ℓ', one can create the transition matrix $(a, \ell) \to (a', \ell')$. Then the stationary distribution of (a, ℓ) can be computed.
4. Using the stationary distribution, compute the sums K and L. Compare the resulting K/L with the guess in step 1.
5. If the guess was correct, we found an equilibrium. If not, revise the guess and repeat from step 1 until convergence.

See Mukoyama (2019) and Mukoyama (2020) for details.

Guess w and r ← $\left(\frac{K}{L}\right)$ guess

- Perform optimization

$$V(a,l) = \max u(c) + \beta E[V(a',l') \mid l]$$

s.t. ...

$$a' = a'(a,l) \qquad \Pi_{ll'}$$

- Stationary distribution

$$\Rightarrow \int a(i)\,di = K$$

$$\int l(i)\,di = L$$

new $\frac{K}{L}$ compare

update $\frac{K}{L}$

repeat until guess $\frac{K}{L} \approx$ new $\frac{K}{L}$

How to compute the stationary distribution

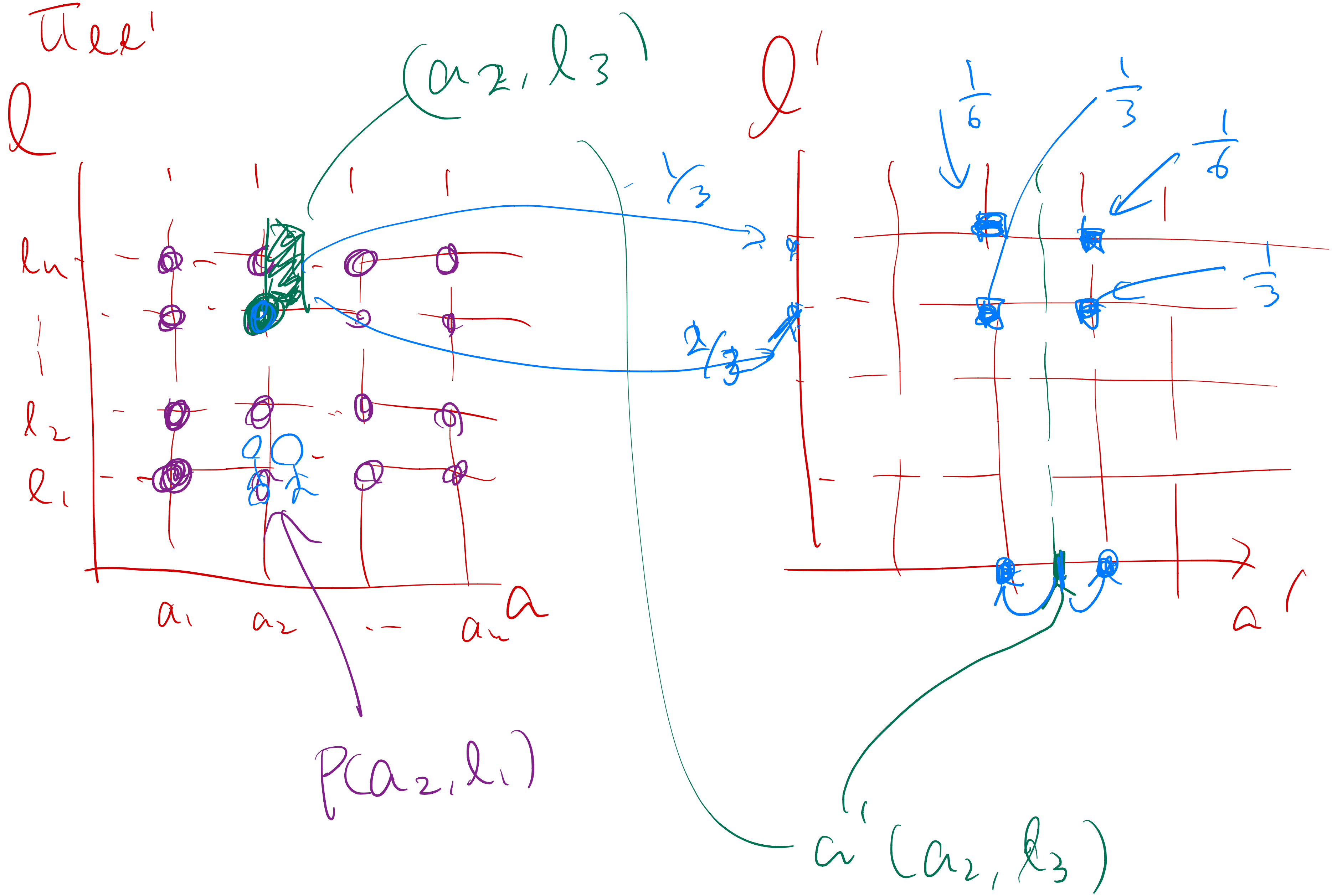
Given the decision rule $a' = a'(a, \ell)$ and the transition probability $\pi_{\ell\ell'}$, how do we compute the matrix (or vector) of populations on the a, ℓ grids, $\{p(a_i, \ell_j)\}$ (where $\sum_i \sum_j p(a_i, \ell_j) = 1$), that is invariant? There are multiple approaches.

1. Simulate. Start from some (a, ℓ). Obtain a' from the decision rule (choose i where a' is closest), generate a random number and obtain ℓ'. Simulate many people.
2. Think of $p(a_i, \ell_j)$ as a histogram. Move $\pi_{\ell_j \ell_k}$ fraction (that is, $\pi_{\ell_j \ell_k} p(a_i, \ell_j)$ units) to $(a'(a_i, \ell_j), \ell_k)$ grid. If $a'(a_i, \ell_j)$ is not on the grid, distribute appropriately. (For example, if $a'(a_i, \ell_j)$ is in the midpoint of a_m and a_{m+1}, give half to a_m and half to a_{m+1}.) Repeat until $p(a_i, \ell_j)$ settle. See Heer and Maußner (2005) and Young (2010) for details.
3. Use the matrix representation $\mathbf{p} = \mathbf{A}\mathbf{p}$ and find $\mathbf{p}$ that satisfy $(\mathbf{A} - \mathbf{I})\mathbf{p} = \mathbf{0}$.

I usually use the second one.

$a' = a'(a, l)$

$\pi_{ll'}$



Policy experiment (Mukoyama, 2020)

Let us work with a policy experiment. Here, think of a wealth tax—the government take away τa_t from the consumers (where $\tau > 0$) and transfer back in lump-sum manner (the transfer is T_t).

- Before time 0, assume that the consumers believe that the tax is 0 forever.
- At time 0, suddenly and unexpectedly (an "MIT shock"), the tax switches to a positive number τ, and it is announced that it will stay at τ forever.
- The new problem:

$$V_t(a_t, \ell_t) = \max_{c_t, a_{t+1}} U(c_t) + \beta E[V_{t+1}(a_{t+1}, \ell_{t+1})|\ell_t]$$

subject to

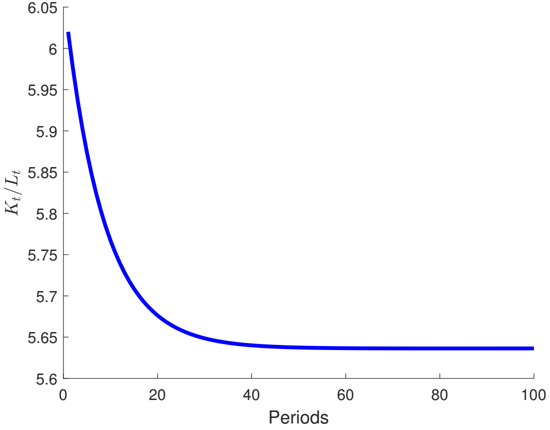
$$c_t + a_{t+1} = w_t \ell_t + (1 + r_t - \tau_t - \delta)a_t + T_t$$

and

$$a_{t+1} \geq b.$$

Policy experiment

- Starting from time 0, K_t/L_t experiences a gradual transition, implying that the prices w_t and r_t are time-dependent.



Computation

To compute the transition dynamics, take the following steps.

1. Compute the steady states of $\tau_t = 0$ (initial steady state) and $\tau_t = \tau$ (final steady state) separately.
2. Assume that the economy is in the final steady state at sufficiently far future (e.g. $t = 100$).
3. Guess the path of K_t/L_t for $t = 1, ..., 99$.
4. Starting from the final steady state, solve the optimization **backwards**. That is:
   4.1 Using the value function of the final steady state as $V_{100}(a, \ell)$ (and using the w_{99} and r_{99} from the guessed K_{99}/L_{99}), solve the Bellman equation for $V_{99}(a, \ell)$.
   4.2 Using the $V_{99}(a, \ell)$, solve the Bellman equation for $V_{98}(a, \ell)$. Go back until $V_0(a, \ell)$. Record all decision rules.
5. Starting from the initial steady state and using above decision rules, compute the distribution of (a_t, ℓ_t) **forward**, for $t = 1, 2, ..., 99$. (Use simulation or histograms.)
6. Compute K_t/L_t from the above (a_t, ℓ_t). Compare with the guess. If the guess was wrong, update and repeat.

Step #1

① Compute the steady state with

$\tau_t = 0$ (start)

② Compute the steady state with

$\tau_t = \tau > 0$ (end)

new S.S.

Step #2 (transition) Guess

$w_t, r_t, T_t \leftarrow \frac{K_t}{L_t}$ $\Rightarrow$ $\left\{\frac{K_t}{L_t}\right\}_{t=1}^{\infty\;100}$ $\left\{T_t\right\}_{t=0}^{\infty\;100}$

Assume that at $t=100$, the economy is in the final steady state

$$V_{99}(a_t, l_t) = \max U(c_t) + \beta E\left[V_{t+1}(a_{t+1}, l_{t+1}) \mid l_t\right]$$

V_{100}

s.t.

$$a_{t+1} = w_{99} l_t + (1 + r_{99} - \delta) a_t + T_{99}$$

new ξ, ς → V_{99}

$$a'_{99} = a'_{99}(a_{99}, l_{99})$$

$\vdots$

$$a'_0 = \ldots$$

$$V_{98}(a_{98}, l_{98}) = \ldots$$

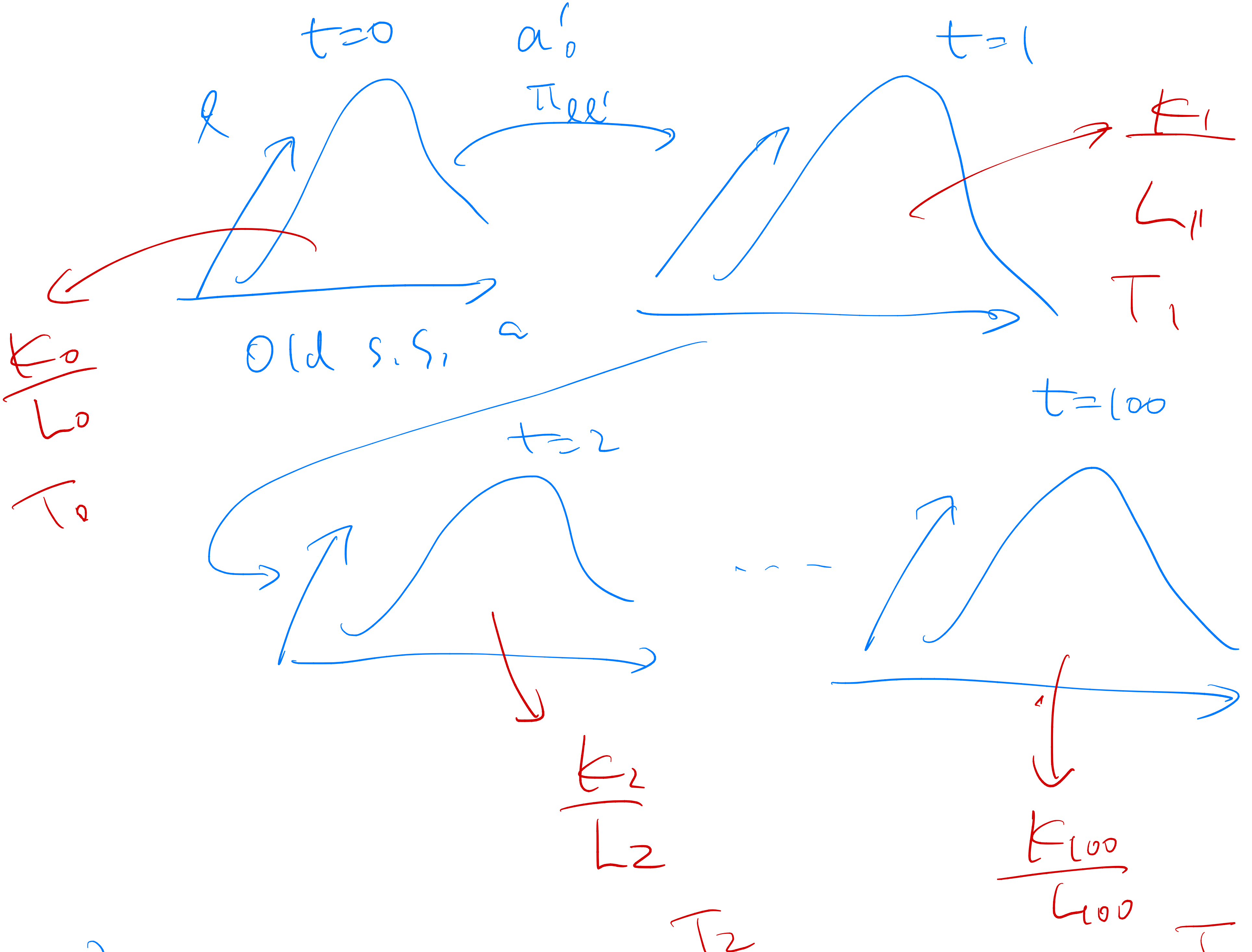
w_{98} r_{98} T_{98}

↓

$$V_0(a_0, l_0) = \ldots$$

w_0 r_0 T_0

→ Compute distribution of (a, l) from $t=0$



⇒ Compare w/ the Guess, repeat

Evaluating the welfare effect of taxation

- One question we might ask is: Are consumers better off with the policy change?
- In this environment, the welfare effects are different across individuals.
- One popularly used metric is the **consumption equivalence**: for consumer i, find $\lambda(i)$ that satisfies

$$E_0\left[\sum_{t=0}^{\infty}\beta^t U((1+\lambda(i))c_t(i)^{\tau_t=0})\right] = E_0\left[\sum_{t=0}^{\infty}\beta^t U(c_t(i)^{\tau_t=\tau})\right].$$

  when $\lambda(i) > 0$, the consumer is better off with the new policy.
- It is easy to compute: for $U(c) = \log(c)$,

$$\lambda(i) = \exp[V^{\tau_t=\tau}(a(i),\ell(i)) - V^{\tau_t=0}(a(i),\ell(i))] - 1.$$

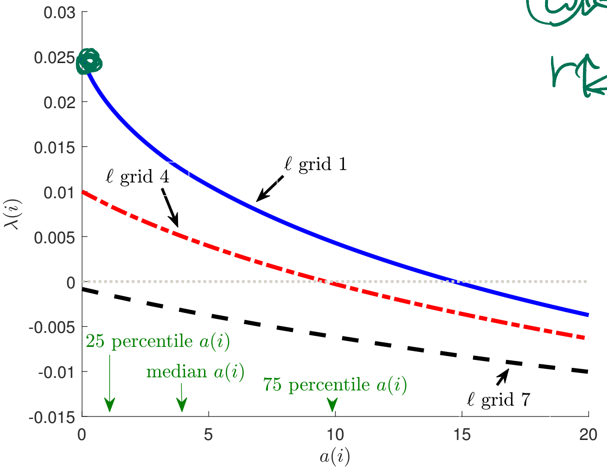
w/ policy τ — w/o policy change ($\tau=0$)

Evaluating the welfare effect of taxation

Below is an example: $U(c) = \log(c)$, $\beta = 0.96$, $\alpha = 0.36$, $\delta = 0.08$, $b = 0$. ℓ_t follows $\log(\ell_t) = \rho \log(\ell_{t-1}) + \sigma(1-\rho^2)^{1/2}\epsilon_t$, where ϵ_t follows $N(0,1)$. $\rho = 0.9$ and $\sigma = 0.4$.

$w \leftarrow$ incr. $\frac{K}{L} \downarrow$

$r \uparrow \leftarrow$ decr. $\frac{K}{L} \downarrow$



Business cycles

- The application to business cycles (aggregate shocks) is very important, not just because business cycles are important, but because over the cycle, the "micro shocks" and "macro shocks" are interacted.
  - Major "micro" labor market events, such as unemployment and job-to-job transitions, are strongly correlated to macro cycles.
  - Firms' entry and exit are also correlated with macro conditions.
- Examples:
  - Costs of business cycles
  - Labor market flows
  - Various policies (including monetary and fiscal policies)

Business cycles

The difficulty is now that the distribution of idiosyncratic state (e.g. the distribution of a) moves around over time. The distribution of a matters for the distribution of a', which determines K'. Knowing K' is important for each consumer, because K is a state variable today (thus K' is the state variable tomorrow) through $r(K/L)$ and $w(K/L)$. Therefore, in principle, the **distribution of** a is the state variable for the consumer. Because the distribution of a is a large object, we need some technique for computation.

- The classic computational method by Krusell and Smith (1998). Assume that the consumer only keep track of a few moments of the distribution. (Skip today)
- Use the deterministic dynamics above (Boppart et al., 2018): explained below
- Many, many techniques recently developed (skip today)

Using MIT shock to compute business cycle statistics

- Think of, for example, a shock to TFP, as in the RBC model. Let the deterministic time-series of the TFP z_t be $\{\bar{z}\exp(\sigma), \bar{z}\exp(\rho\sigma), \bar{z}\exp(\rho^2\sigma)...\}$, where $\bar{z}$ is the steady-state value, σ and ρ are the AR(1) standard deviation and the persistence of shock. Note that the economy will go back to the original steady state, instead of moving to a new steady state.
- One can compute the deterministic transition dynamics following the "backward-forward" method above. Store the values of variable of interest, as log-deviations from the steady-state value. For example, for GDP, $y(0), y(1), y(2), \ldots$.
- Simulate the economy. Draw shocks from $N(0,1)$. Suppose the shock is ϵ_0, ϵ_1, ϵ_2, ... Then the GDP at time t can be computed as

$$Y_t = \bar{Y}\exp\left(\sum_{m=0}^{\infty} y(m)\epsilon_{t-m}\right).$$

(Practically, instead of ∞ take a large finite value.)

Using MIT shock to compute business cycle statistics

Pros and cons:

- **Pro:** Intuitive and simple to implement. Once the transition dynamics is computed, simulation can be done many times, and a larger shock can be accommodated by simply scaling up in the simulation.
- **Pro:** Many different shocks can be accommodated, by just computing the transition dynamics separately, simulating multiple shocks simultaneously, and adding up the response to different shocks (see, for example, Krusell et al., 2020).
- **Pro:** Nonlinear responses to shocks are incorporated.
- **Con:** The effect of uncertainty on the steady-state level is not accounted for.
- **Con:** No obvious way to verify if the approximation by the deterministic sequence (and scaling up and down with ϵ) is accurate.

Recap of the lectures

- Last week, I introduced the basics: the main motivations to work with incomplete-market models and the Huggett model (asset pricing).
- Today, I covered the computation of Aiyagari model, how to work with the transitional dynamics after a policy change, and move ahead to the business cycle analysis.

References

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