Firm Dynamics and the Macroeconomy: Basics

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On firm growth

Two small points first:

- First, note that individual firm growth is not necessary or sufficient for aggregate growth.
- Second, the loss from missing entry can be large if we take firm growth into account.

An example:

- Labor supply is elastic (employment is demand-determined). One firm hires one worker.
- The production of a firm who enters at time τ and age a (today is $t = a + \tau$) is $A_\tau e^{\gamma a}$. $\gamma > 0$ is the firm growth rate. Assume that $A_\tau = A_0 e^{g\tau}$.
- The surviving firms at age a is $e^{-\delta a}$. Assume $\delta > \gamma$.
- The mass of entrants is 1.
- Outcome: The total employment is $\int_0^\infty e^{-\delta a} da = 1/\delta$. The aggregate production is $A_0 e^{gt}/(\delta + g - \gamma)$.
- If Δ units of entrants are lost, the immediate loss is $\Delta A_\tau dt$ but the present value of loss is $\Delta A_\tau/(\rho + \delta - \gamma)$, where ρ is the discount rate.

On firm growth

Derivations:

- Employment (number of firms):

$$L = \int_0^\infty e^{-\delta a} da = \frac{1}{\delta}.$$

- Aggregate output:

$$Y = \int_0^\infty A_\tau e^{\gamma a} e^{-\delta a} da = \int_0^\infty A_0 e^{g(t-a)} e^{\gamma a} e^{-\delta a} da = \frac{A_0 e^{gt}}{\delta + g - \gamma}.$$

- Loss from the lack of entry:

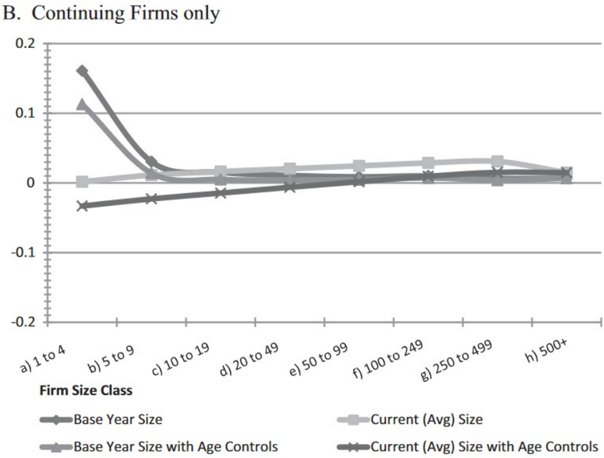
$$\int_0^\infty e^{-\rho t} \Delta A_\tau e^{\gamma a} e^{-\delta a} da = \Delta A_\tau \int_0^\infty e^{-\rho t} e^{\gamma a} e^{-\delta a} da = \frac{\Delta A_\tau}{\rho + \delta - \gamma}.$$

Another note:

- In many endogenous growth models, the driving force of the aggregate growth is entrants' innovations. Or often it is implicitly assumed that entrants keep up with incumbents.

Some facts on firm growth

Figures from Haltiwanger et al. (2013)

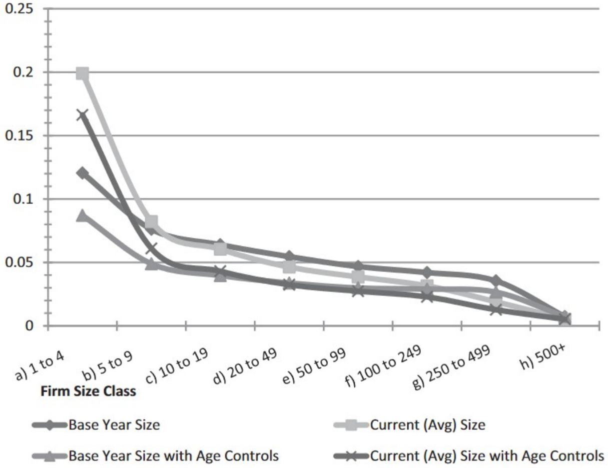


- The growth rate of a firm is independent of size: “Gibrat’s Law” (mixed supports in the data)

Some facts on firm growth

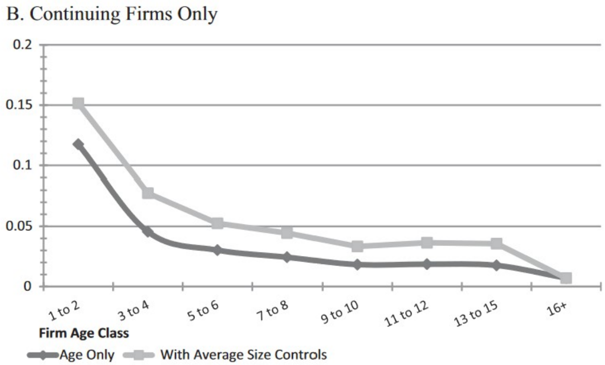
Figures from Haltiwanger et al. (2013)

FIGURE 3.—FIRM EXIT BY FIRM SIZE



Some facts on firm growth

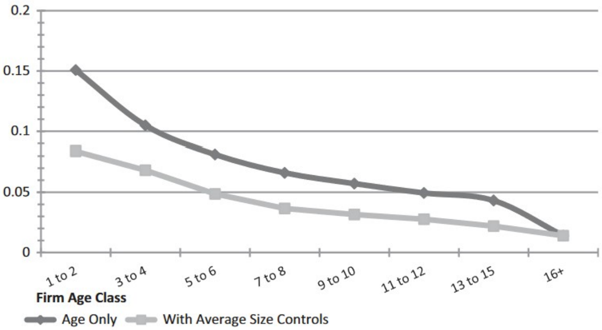
Figures from Haltiwanger et al. (2013)



Some facts on firm growth

Figures from Haltiwanger et al. (2013)

FIGURE 5.—FIRM EXIT BY FIRM AGE



Gibrat's law (with positive growth) implies a Pareto tail

Let us go back to the previous example, but with growth in terms of size.

- A firm is born with size $A > 0$.
- It grows at the rate γ.
- It exits at the rate δ.
- At the stationary distribution, for any size $x > A$, the density $s(x)$ has to satisfy

  $$s(xe^{\gamma dt})\Delta e^{\gamma dt} = e^{-\delta dt} s(x)\Delta$$

  for small dt and Δ.
- Guess that the distribution is Pareto: $s(x) = Fx^{-(\kappa+1)}$ (for $x \geq A$), where $F > 0$ and κ is the shape parameter. Then

  $$F(xe^{\gamma dt})^{-(\kappa+1)} e^{\gamma dt} = e^{-\delta dt} Fx^{-(\kappa+1)}$$

  and therefore

  $$\kappa = \frac{\delta}{\gamma}.$$

  A large γ or a small δ implies a small κ (thick tail).

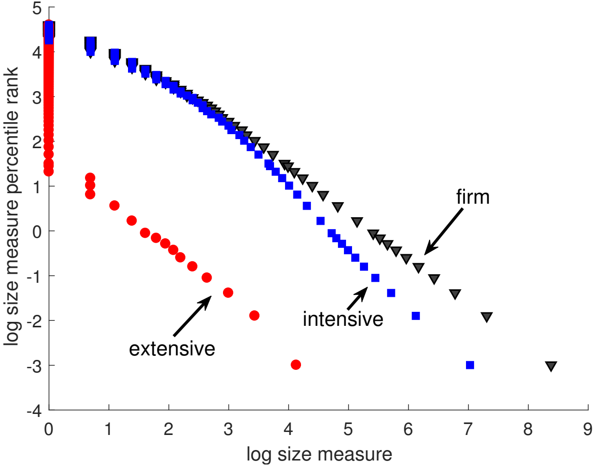
Gibrat's law (with positive growth) implies a Pareto tail

$$s(xe^{\gamma dt})\Delta e^{\gamma dt} = e^{-\delta dt} s(x)\Delta$$

The distributions in the US, from QCEW

From Cao et al. (2020):

- Intensive margin: average employment per establishment in each firm
- Extensive margin: number of establishments in each firm



US versus Japan

Tables from Mukoyama (2009)

Table 2 Entry and Exit Rates

Annual, percent

	United States	Japan
Entry rate	11.6	4.4
Exit rate	10.2	4.4

Table 4 Establishments in the United States and Japan: Average Sizes

	United States	Japan
Average size of all establishments	17.6	9.4
Average size of opening establishments	8.3	9.6
Average size of closing establishments	9.0	7.9

- Low average size in Japan (despite a large entrant size) with low exit rates: lack of growth in establishments.

A simplified version of Mukoyama and Osotimehin (2019)

One way of looking at this paper is Hopenhayn and Rogerson (1993) with endogenous productivity shocks (and growth).

- Representative consumer:

$$\mathbf{U} = \sum_{t=0}^{\infty} \beta^t [\log(C_t) - \xi L_t],$$

  where $\beta \in (0, 1)$ and $\xi > 0$.

- Final good (used for consumption and R&D):

$$Y_t = \left(\int_{\mathcal{N}_t} {q_{jt}}^{\psi} {y_{jt}}^{1-\psi} dj \right)^{\frac{1}{1-\psi}}.$$

- Quality q_{jt} can be improved by incumbent intermediate producer's innovation or entrants' creative destruction ("quality ladders").

- Aggregate quality (productivity) index: $Q_t \equiv \bar{q}_t^{\frac{\psi}{1-\psi}}$.

Intermediate-good firms (monopolistic competition)

- Good j only produced by the cutting-edge producer (monopoly).
- Produced only by labor.

$$y_{jt} = \ell_{jt}.$$

- Exit if entrant innovates on product j or if hit by an exogenous shock δ.
- Firing costs
  - Tax for each worker fired τw.
  - The tax is transferred lump-sum to the consumer.
- Incumbents can innovate on their own products.
- Entrants innovate randomly across different products.
- Innovation is stochastic.

$$q_{jt} = \begin{cases} (1+\lambda_i) q_{j,t-1} & \text{if innovates} \\ q_{j,t-1} & \text{if does not innovate} \end{cases}$$

where $i = I, E$.

Quality ladders

Model: innovation

- Incumbents
  - Improves the quality of its own product by R&D (in final goods).
  - Probability of successful innovation: x_{Ijt}
  - Innovation cost:

$$\mathbf{r}_{Ijt} = \theta_I Q_t \frac{q_{jt}}{\bar{q}_t} {x_{Ijt}}^{\gamma}.$$

- Entrants
  - First pay the entry cost ϕQ_t (become a potential entrant) and then conduct R&D
  - Probability of successful innovation: x_{Et}
  - Innovation cost:

$$\mathbf{r}_{Et} = \theta_E Q_t {x_{Et}}^{\gamma}.$$

  - Cost is increasing in the aggregate productivity Q_t.
  - Creative destruction rate: $\mu = m x_E$, where m is the number of potential entrants.

Model: firm's problem

$$
\begin{aligned}
&V_t(q_t, \ell_{t-1}) \\
&\qquad = \max_{\ell_t, x_{It}} \quad \Pi_t(q_t, \ell_{t-1}, \ell_t, x_{It}) \\
&\qquad\qquad + \frac{1}{1+r} \Big\{ (1-\mu_t) \left[(1 - x_{It}) Z_{t+1}(q_t, \ell_t) \right. \\
&\qquad\qquad\qquad \left. + x_{It} Z_{t+1}((1+\lambda_I) q_t, \ell_t) \right] \\
&\qquad\qquad\qquad - \mu_t \tau w_{t+1} \ell_t \Big\}
\end{aligned}
$$

where

$$
Z_t(q_t, \ell_{t-1}) = (1-\delta) V_t(q_t, \ell_{t-1}) - \delta \tau w_t \ell_{t-1}
$$

and

$$
\begin{aligned}
&\Pi_t(q_t, \ell_{t-1}, \ell_t, x_{It}) \\
&\qquad = (p_t - w_t)\ell_t - \theta_I Q_t \frac{q_t}{\bar{q}_t} {x_{It}}^{\gamma} - \tau w_t \max\langle 0, \ell_{t-1} - \ell_t \rangle,
\end{aligned}
$$

with $p_t = q_t^{\psi} {y_{jt}}^{-\psi} {Y_t}^{\psi}$.

Model: entry

- Free-entry condition

$$\max_{x_{E_t}} \left\{ -\theta_E Q_t {x_{Et}}^{\gamma} - \phi Q_t + \frac{1}{1+r} x_{Et} \bar{V}_{E,t+1} \right\} = 0,$$

- $\bar{V}_E$ is the expected value of entry
- Creative destruction rate $\mu = m x_E$

Model: solving for the stationary equilibrium

1. Normalized model: $\hat{Y} \equiv Y_t/Q_t$, $\hat{w} \equiv w_t/Q_t$, $\hat{q} \equiv q_t/\bar{q}_t$,....
2. Given g_q, μ, $\hat{Y}$, $\hat{w}$
   - Compute value functions and decision functions
   - Stationary distribution of firms over $\hat{q}$, α and ℓ_{-1}
3. Stationary GE conditions: find g_q, μ, $\hat{Y}$, $\hat{w}$ such that
   (i) $\hat{Y}$ consistent with firms' output decision
   (ii) $\hat{\bar{V}}_E$ satisfies the free entry condition
   (iii) $\hat{Y} = \hat{C} + \hat{R}$
   (iv) $\frac{1}{N} \int \int \hat{q} f(\hat{q}, \ell) d\ell dq = 1$

The steps are very similar to the standard firm dynamics model (Hopenhayn and Rogerson, 1993) and the standard heterogeneous-agent models (Bewley-Huggett-Aiyagari)

Model: stationarized problem

$$\hat{V}(\hat{q},\ell) = \max_{\ell' \geq 0, x_I} \hat{\Pi}(\hat{q},\ell,\ell',x_I) + \beta\left((1-\mu)\hat{S}(x_I,\hat{q}/(1+g_q),\ell') - \mu\tau\hat{w}\ell'\right),$$

where

$$\hat{S}(x_I,\hat{q}/(1+g_q),\ell') = (1 - x_I\hat{Z}(\hat{q}/(1+g_q),\ell') + x_I\hat{Z}((1+\lambda)\hat{q}/(1+g_q),\ell').$$

$$\hat{\Pi}(q,\alpha,\ell,\ell',x_I) = (\hat{q}^{\psi}\ell'^{-\psi}\hat{Y}^{\psi} - \hat{w})\ell' - \theta_I\hat{q}{x_I}^{\gamma} - \tau\hat{w}\max\langle 0,\ell-\ell'\rangle.$$

- The frictionless case can be solved analytically. (Next slide)
- For the case with $\tau > 0$, there is one more step to computation: rewrite ℓ as the deviation from the frictionless level of ℓ (which can be computed from static optimization). This step is important because ℓ can have a very long tail.

Model: frictionless benchmark

- The value function $\hat{Z}$ is linear in productivity q.
- The innovation decision is independent of q ($\rightarrow$ Gibrat's law)
- Note: the normalized productivity next period is $(1+\lambda_I)\hat{q}/(1+g_q)$ with successful innovation, but without innovation $\hat{q}/(1+g_q) < \hat{q}$. $\rightarrow$ the firm has to contract if it does not innovate.
- Right tail of the productivity distribution is Pareto

$$F(\hat{q} > u) \propto u^{-\kappa},$$

where κ is the solution to:

$$1 = (1-\delta)\left[(\mu + (1-\mu)x_I)\gamma_i^{\kappa} + (1-\mu-(1-\mu)x_I)\gamma_n^{\kappa}\right].$$

- Growth rate of $\bar{q}$:

$$1 + g_q = (1-\delta)[(1+\lambda_I x_I)(1-\mu) + (1+\lambda_E)\mu] + \delta(1+\lambda_E)\bar{q}^h$$

Results: the effect of a higher firing tax

Innovation rate of entrants is lower.

- period profit (tax payment/distortion/wages) $(-)$

Innovation rate of incumbents can be higher or lower.

- period profit (tax payment/distortion/wages) $(-)$
- creative destruction effect (lower μ) $(+)$
- tax-escaping effect (escape tax payment by innovating) $(+)$

Klette and Kortum (2004)

- Klette and Kortum (2004) is a standard reference in talking about firm dynamics and endogenous growth.
- Compared to the classic quality-ladder models, it adds the element of incumbent firms growing by adding more products by innovating on other firms' products.
- Some shortcomings:
  - Cannot generate a fat tail of firm size distribution
  - Do not allow firms to come up with entirely new products. (related to the first point)
  - Do not allow the leaders to improve their own products. (also related to the first point)
  - See the discussion in Cao et al. (2020).

Klette and Kortum (2004): Quality ladders

- Continuum of differentiated goods, each good indexed by $j \in [0, 1]$.
- The quality of good j at generation k: $z(j, k)$. One generation of good j is produced by one firm.
- The size of ladder:

$$q(j, k) \equiv \frac{z(j, k)}{z(j, k-1)}.$$

- One unit of good is produced by one unit of labor. Thus the unit cost is w.

Klette and Kortum (2004): Goods demand

- Utility
$$U = \int_0^\infty e^{-\rho t} \ln(C_t) dt,$$
where
$$\ln(C_t) = \int_0^1 \left[\ln \sum_{k=-1}^{J_t(j)} x_t(j,k) z(j,k) \right] dj.$$
- Normalizing the income (=expenditure) at each t to 1, the demand for good j is unit elastic (after intra-temporal optimization):
$$\sum_{k=-1}^{J_t(j)} p_t(j,k) x_t(j,k) = 1.$$

Klette and Kortum (2004): Firm's optimization

- Because the unit cost is constant (and the same across firms) and the demand is unit elastic, the firm with the best quality tries to maximize monopoly profit by limit pricing:

$$p_t(j) = q_t(j)w.$$

- The profit is

$$\pi_t(j) = [p_t(j) - w]x_t(j) = 1 - \frac{1}{q_t(j)}.$$

  When $q_t(j)$ is constant at q, $\pi_t(j)$ is constant at π and $x_t(j)$ is constant at $1/wq$.

- A firm is defined as the collection of products it produces. By innovating, a firm can add another product to the products it already produces. It may lose products randomly (because of other firms' innovation). A firm that produces n products earn $n\pi$.

Klette and Kortum (2004): Firm dynamics

- The firm size n can increase by innovation. Innovation cost function is

  $$w\mathcal{L}_R = C(I, n),$$

  where $\mathcal{L}_R$ is the R&D input (in labor), I is the Poisson probability of innovation. C function is homogeneous of degree one. Thus

  $$w\frac{\mathcal{L}_R}{n} = C\left(\frac{I}{n}, 1\right).$$

  Let $\lambda = I/n$ and rewrite this as

  $$w\ell_R(\lambda) = c(\lambda).$$

- The firm loses one product with probability μn.

Klette and Kortum (2004): Bellman equation

- The HJB equation of a firm is

$$rV(n) = \max_I \{\pi n - C(I, n) + I[V(n+1) - V(n)] \\ -\mu n[V(n) - V(n-1)]\}.$$

- Guessing $V(n) = vn$ and $I(n) = \lambda n$,

$$rv = \max_\lambda \{\pi - c(\lambda) + \lambda[v(n+1) - vn] \\ -\mu[vn - v(n-1)]\}.$$

  Thus v and λ solve

$$c'(\lambda) = v$$

  and

$$rv = \pi - c(\lambda) + \lambda v - \mu v.$$

Klette and Kortum (2004): Entry and exit

- A firm exits when it has no product to produce.
- A firm can enter by paying a fixed cost $F = wh$, where h is the unit of labor that is necessary for entry. We assume free entry:
$$F = v (= wh).$$
- The entrants' innovation rate is η: the labor used for startup is therefore $L_s = \eta h$.

Klette and Kortum (2004): General equilibrium in balanced growth

- Total labor supply L can be distributed to L_x: production, L_R: R&D, and L_s: startup.
- Because $v = wh$ from free entry and the optimal R&D $v = w\ell'_R(\lambda)$, λ can be solved from $h = \ell'_R(\lambda)$. Given this λ, $L_R = \ell_R(\lambda)$ is a constant.
- From the normalization, $r = \rho$ and also the budget constraint is

  $$1 = wL + rv = wL + \rho wh.$$

  Thus $w = 1/(L + \rho h)$. Therefore

  $$L_x = \int x(j)dj = \int \frac{1}{qw}dj = \frac{1-\pi}{w} = \frac{1-\pi}{L+\rho h}.$$

  This is also a constant number.
- Thus, $L_s = L - L_x - L_R$ can be used to solve η by

  $$\eta = \frac{1}{h}\left(L - \frac{1-\pi}{L+\rho h} - \ell_R(\lambda)\right).$$

Klette and Kortum (2004): Properties

In addition to generating endogenous growth, the model allows us to calculate various statistics that can be compared to the data:

- Firm size and subsequent exit probability
- The average longevity of firm in relation to age
- The expected size of firm for a given age
- The size distribution of firms.

Three key numbers: λ (incumbents' innovation), η (entrants' innovation), and μ (total innovation). $\mu = \lambda + \eta$.

- The firm growth follows the Gibrat's law (kind of): expected growth rate is $\lambda - \mu = -\eta$.
- The stationary distribution does not have a Pareto tail.

$$\text{mass at size } n = \text{const.} \times \frac{1}{n} \left(\frac{\eta}{\mu} \right)^n .$$

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