

# Relationship between Maximum Principle and Dynamic Programming

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August 2023

## 1 The saving problem

In this note, we demonstrate the relationship between continuous-time Maximum Principle (optimal control) and continuous-time Dynamic Programming using a simple saving problem. The problem is

$$\max_{c(t), a(t)} \int_0^{\infty} e^{-\rho t} u(c(t)) dt$$

subject to

$$\dot{a}(t) = ra(t) - c(t),$$

where  $c(t)$  is consumption at time  $t$ ,  $a(t)$  is asset at time  $t$ ,  $\rho > 0$  is the discount rate,  $r > 0$  is the interest rate, and  $u(\cdot)$  is an increasing and concave utility function.  $\dot{a}(t) = da(t)/dt$  represents the time derivative.

## 2 Maximum Principle

For optimization using Maximum Principle, two alternative formulations can be used. The first method uses the *present-value Hamiltonian*:

$$H^p(t) \equiv e^{-\rho t} u(c(t)) + \lambda(t)(ra(t) - c(t)),$$

where  $\lambda(t)$  is the costate variable. The first-order conditions for this formulation are

$$\frac{\partial H^p(t)}{\partial c(t)} = 0$$

and

$$\frac{\partial H^p(t)}{\partial a(t)} + \dot{\lambda}(t) = 0.$$

(The necessary condition for optimality includes the transversality condition, but we omit it here.)

The second uses the *current-value Hamiltonian*:

$$H^c(t) = u(c(t)) + \mu(t)(ra(t) - c(t)),$$

where  $\mu(t)$  is the costate variable in this case. The first-order conditions are

$$\frac{\partial H^c(t)}{\partial c(t)} = 0 \quad (1)$$

and

$$\frac{\partial H^c(t)}{\partial a(t)} + \dot{\mu}(t) - \rho\mu(t) = 0. \quad (2)$$

It is easy to check that both approaches are equivalent, with the relationship  $\lambda(t) = e^{-\rho t}\mu(t)$ . Therefore, we will work with the current-value Hamiltonian below. For the saving problem above, the first-order condition (1) is

$$u'(c(t)) = \mu(t) \quad (3)$$

and (2) is

$$r\mu(t) + \dot{\mu}(t) - \rho\mu(t) = 0. \quad (4)$$

### 3 Dynamic Programming

We will start by defining the value function

$$V(a(t)) = \int_t^\infty u(c^*(t))dt,$$

where  $c^*(t)$  is the optimal value of  $c(t)$ . To (heuristically) derive the first-order condition, we start by considering a discrete-time formulation with period length  $\Delta$ . Later we will take the limit of  $\Delta \rightarrow 0$ . The discrete-time Bellman equation is

$$V(a(t)) = \max_{c(t), a(t+\Delta)} u(c(t))\Delta + \frac{1}{1 + \rho\Delta}V(a(t + \Delta)) \quad (5)$$

subject to

$$a(t + \Delta) = (1 + r\Delta)a(t) - c(t)\Delta.$$

The first-order condition is

$$u'(c(t))\Delta = \frac{1}{1 + \rho\Delta}V'(a(t + \Delta))\Delta.$$

Dividing both sides by  $\Delta$  and take  $\Delta \rightarrow 0$ , we obtain

$$u'(c(t)) = V'(a(t)). \quad (6)$$

The Bellman equation (5) can be rewritten as (with optimally chosen  $c(t)$  and  $a(t)$ )

$$\frac{V(a(t + \Delta)) - V(a(t))}{\Delta} + u(c(t)) - \frac{\rho}{1 + \rho\Delta}V(a(t + \Delta)) = 0.$$

Taking the limit of  $\Delta \rightarrow 0$ ,

$$V'(a(t))\dot{a}(t) + u(c(t)) - \rho V(a(t)) = 0. \quad (7)$$

This equation is the *Hamilton-Jacobi-Bellman (HJB) equation* and can be interpreted as an “asset equation,” that is, rewriting

$$\rho V(a(t))dt = u(c(t))dt + V'(a(t))\dot{a}(t)dt,$$

the left-hand side is the required return of an asset that has the value of  $V(a(t))$  over the time period  $dt$ , and the right-hand side is the income gain  $u(c(t))dt$  plus the capital gain  $V'(a(t))\dot{a}(t)dt$ .

Replacing  $\dot{a}(t)$  in (7) by  $ra(t) - c(t)$ , we obtain

$$rV'(a(t))a(t) - V'(a(t))c(t) + u(c(t)) - \rho V(a(t)) = 0.$$

This equation has to hold for all  $a(t)$ , and thus we can take the derivative with respect to  $a(t)$  on both sides:

$$rV''(a(t))a(t) + rV'(a(t)) - V''(a(t))c(t) - V'(a(t))\frac{dc(t)}{da(t)} + u'(c(t))\frac{dc(t)}{da(t)} - \rho V'(a(t)) = 0.$$

Rearranging using  $\dot{a}(t) = ra(t) - c(t)$  and (6),

$$rV'(a(t)) + V''(a(t))\dot{a}(t) - \rho V'(a(t)) = 0. \quad (8)$$

Now we can see the correspondence between the Maximum Principle equations ((3) and (4)) and the Dynamic Programming equations ((6) and (8)) with  $\mu(t) = V'(a(t))$ .