Explaining the Poisson Process

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January 2024 (updated April 2024)

Consider a Bernoulli trial that can result in a "success" or a "failure." The situation can be finding a job, losing a job, succeeding in innovation, etc. Suppose that the probability of success is $\lambda \in [0, 1]$ for one trial.

Suppose that during one unit of time (say, one year), one trial is made. From the above assumptions, the expected number of successes during this one year is λ. Now, suppose that we divide the period into two and have one trial every six months. Each trial is independent. If we adjust the success probability of each trial to $\lambda/2$, the expected total number of successes during the one year is still λ. In general, if we divide the period (one year) into n subperiods and make the success probability λ/n in each subperiod, we still keep the expected total number of success λ, although now we may have many successes during one year.

With n trials, the distribution of the number of successes follows a binomial distribution

$$b(k, n) = \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}, \tag{1}$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

is the binomial coefficient (the number of different ways one can have k successes out of n trials). $b(k, n)$ represents the probability of k successes during this time period.

Note that, in (1),

$$b(0, n) = \left(1 - \frac{\lambda}{n}\right)^n \tag{2}$$

and

$$\frac{b(k, n)}{b(k-1, n)} = \frac{n-(k-1)}{k} \frac{\lambda/n}{1 - \lambda/n} = \frac{\lambda - (k-1)\lambda/n}{k - k\lambda/n} \tag{3}$$

for $k \geq 1$ hold. Let us consider a situation where $n \to \infty$. That is, the length of each subperiod approaches zero. In (2),

$$p(0) \equiv \lim_{n\to\infty} b(0, n) = e^{-\lambda}.$$

The second inequality follows from the definition of e. Taking the limit of (3),

$$\lim_{n\to\infty} \frac{b(k, n)}{b(k-1, n)} = \frac{\lambda}{k}$$

holds. Applying this formula for $k = 1, 2, 3, ...$ sequentially, we obtain

$$p(k) = e^{-\lambda} \frac{\lambda^k}{k!}.$$

This $p(k)$ represents the probabilities for the *Poisson distribution*. The *Poisson process* is a stochastic process where the distribution of the number of successes between any time interval $(t, t+T]$ (for $T > 0$) follows a Poisson distribution. The expected number of successes during the time interval $(t, t+T]$ is λT, and we saw that the Poisson process in continuous time can be understood as the limit of the discrete-time situation where repeated independent Bernoulli trials are conducted in every small time interval $\Delta = T/n$, where $\Delta \to 0$. Each trial has the probability of success $\lambda\Delta$ and the total number of trials are $n = T/\Delta$, thus keeping the expected number of successes as λT during the entire time period. Because the probability of success in each trial $\lambda\Delta \to 0$ and the total number of trials $n = T/\Delta \to \infty$ as $\Delta \to 0$, successes occur "infrequently" compared to the total number of trials.

A few notes are in order. First, from the construction, it can be seen that the outcomes during $(t, t+T]$ and during $(t+T+s, t+T+s+S]$ are independent of each other for any $s \geq 0$ and $S \geq 0$. Thus, the Poisson process is a memoryless process. Second, the initial restriction $\lambda \leq 1$ is not necessary; we can always start from a small enough time interval (a large enough n) such that $\lambda/n \leq 1$ and proceed with the above construction.

References

[1] Feller, W (1968). *An Introduction to Probability Theory and Its Applications, 3rd ed.* John Wiley and Sons.