Explaining the RBC Dynamics using the Continuous-Time Ramsey Model

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1 No labor-leisure decision

This note works out the intuition of the real business cycle dynamics using the continuous-time Ramsey model. In particular, we interpret booms and recessions as temporary technology shocks that are unanticipated ("MIT shocks"). First, in this section, we abstract from labor-leisure choice of consumers and focus on the propagation of shocks through capital accumulation.

As the market equilibrium is Pareto efficient in the following economy, we consider the benevolent social planner's problem. Suppose that the household utility is

$$U = \int_0^\infty e^{-\rho t} \log(c(t)) dt,$$

where $c(t)$ is consumption at time t, and $\rho > 0$. The feasibility constraint is

$$\dot{k}(t) = z(t)k(t)^\alpha - \delta k(t),$$

where $z(t)$ is the total factor productivity (TFP) and $k(t)$ is capital stock. α is a parameter between 0 and 1, and $\delta > 0$ is the depreciation rate. The standard procedure for dynamic optimization leads to the Euler equation

$$\frac{\dot{c}(t)}{c(t)} = \alpha z(t)k(t)^{\alpha-1} - (\delta + \rho).$$

The dynamics of the model is described in Figure 1. The red curve is the saddle path that $c(t)$ and $k(t)$ follow during the transition dynamics.

Suppose that the economy is in the steady state with constant $z(t) = z$. Now, consider an unforeseen positive technology shock. At time t_0, $z(t)$ temporarily increases to $z' > z$, during the time period between t_0 to t_1. This temporary shock is not anticipated by the consumers and firms before t_0 (an "MIT shock"), but the consumers and firms have perfect foresight from the time t_0 on. In particular, they are aware that $z(t)$ will go back to z after t_1. The new dynamics is analyzed in Figure 2. In the phase diagram, the $\dot{k}(t) = 0$ curve switches to the blue curve between t_0 and t_1. The $\dot{c}(t) = 0$ line shifts to the right. The consumption

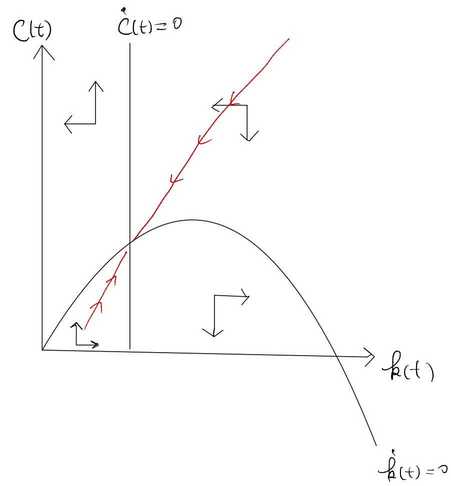

Figure 1: phase diagram

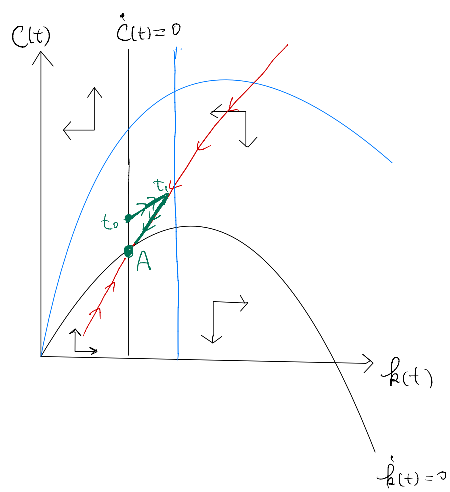

Figure 2: phase diagram

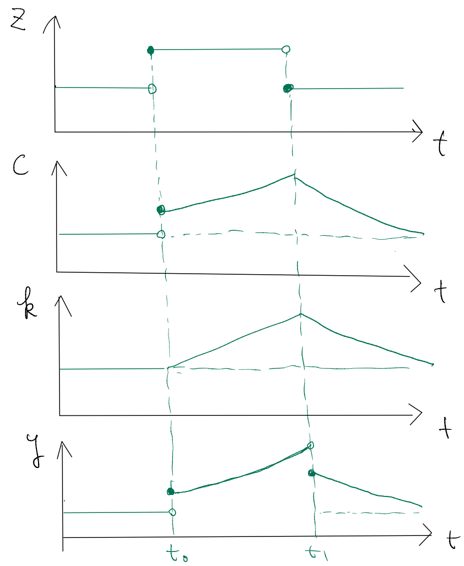

Figure 3: time series of variables

jumps from point A at time t_0, so that at time t_1, the combination of $c(t_1)$ and $k(t_1)$ is on the red saddle path.

At t_0, consumption jumps up. Between t_0 and t_1, the consumers enjoy a higher consumption level with an increasing profile (it is possible to start declining before time t_1), and they also accumulate capital. After t_1, consumption starts declining (if it had not already), gradually converging to the original level. Meanwhile, the consumers decumulate the asset and $k(t)$ converges to the original level. Figure 3 describes the time series of relevant variables. The time series of $y(t)$ reflects the dynamics of both $z(t)$ and $k(t)$.

We can see from the time series of $y(t)$ that there are impulse and propagation. The initial jump of $y(t)$ is due to the impulse of increase in $z(t)$. Then $k(t)$ plays a role in propagation. Net investment (slope of $k(t)$) jumps up at t_0 and jumps down at t_1 to a negative level, and eventually converges back to zero. We can see the comovement of $y(t)$, $c(t)$, and net investment $\dot{k}(t)$. Consumption is smoother than GDP: some of the increase in income is saved, so that the asset (capital) can be used for the extra consumption after $z(t)$ goes back to the original level.

2 With labor-leisure decision

In this section, we consider the same setting, but with the labor-leisure decision. The production function is now

$$y(t) = z(t)k(t)^{\alpha}n(t)^{1-\alpha}.$$

Let the momentary utility function be

$$u(c(t), n(t)) = \log(c(t)) - \xi \frac{n(t)^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}},$$

where $\xi > 0$ and $\eta > 0$. The static first-order condition is

$$\xi n(t)^{\frac{1}{\eta}} = \frac{w(t)}{c(t)},$$

where $w(t)$ is the marginal product of capital, corresponding to wages in the market economy. Thus,

$$\xi n(t)^{\frac{1}{\eta}} = \frac{1}{c(t)}(1-\alpha)z(t)\left(\frac{k(t)}{n(t)}\right)^{\alpha},$$

which can be solved as

$$n(t) = \left(\frac{(1-\alpha)z(t)k(t)^{\alpha}}{\xi c(t)}\right)^{\frac{\eta}{1+\alpha\eta}}.$$

Therefore, the production function can be rewritten as

$$y(t) = z(t)k(t)^{\alpha}\left(\frac{(1-\alpha)z(t)k(t)^{\alpha}}{\xi c(t)}\right)^{\frac{(1-\alpha)\eta}{1+\alpha\eta}} = \gamma z(t)^{\frac{1+\eta}{1+\alpha\eta}} k(t)^{\frac{(1+\eta)\alpha}{1+\alpha\eta}} c(t)^{-\frac{(1-\alpha)\eta}{1+\alpha\eta}},$$

where

$$\gamma \equiv \left(\frac{1-\alpha}{\xi}\right)^{\frac{(1-\alpha)\eta}{1+\alpha\eta}}.$$

From the intertemporal optimization, we can derive a similar Euler equation as in the previous section.

$$\frac{\dot{c}(t)}{c(t)} = \alpha z(t)k(t)^{\alpha-1}n(t)^{1-\alpha} - (\delta+\rho) = \alpha\gamma z(t)^{\frac{1+\eta}{1+\alpha\eta}}(k(t)c(t)^{\eta})^{-\frac{1-\alpha}{1+\alpha\eta}} - (\delta+\rho).$$

The dynamics of capital is dictated by

$$\dot{k}(t) = \gamma z(t)^{\frac{1+\eta}{1+\alpha\eta}} k(t)^{\frac{(1+\eta)\alpha}{1+\alpha\eta}} c(t)^{-\frac{(1-\alpha)\eta}{1+\alpha\eta}} - \delta k(t) - c(t).$$

For simplicity, consider the example: $\delta = 0$, $\alpha = 1/3$, and $\eta = 1$:

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{3}\gamma z(t)^{\frac{3}{2}}(k(t)c(t))^{-\frac{1}{2}} - \rho$$

$$\dot{k}(t) = \gamma z(t)^{\frac{3}{2}} k(t)^{\frac{1}{2}} c(t)^{-\frac{1}{2}} - c(t)$$

$$n(t) = \gamma^{\frac{3}{2}} z(t)^{\frac{3}{4}} k(t)^{\frac{1}{4}} c(t)^{-\frac{3}{4}}.$$

The first two differential equations can be drawn as the phase diagram in Figure 4. This time, the $\dot{c}(t) = 0$ curve is not a straight line.

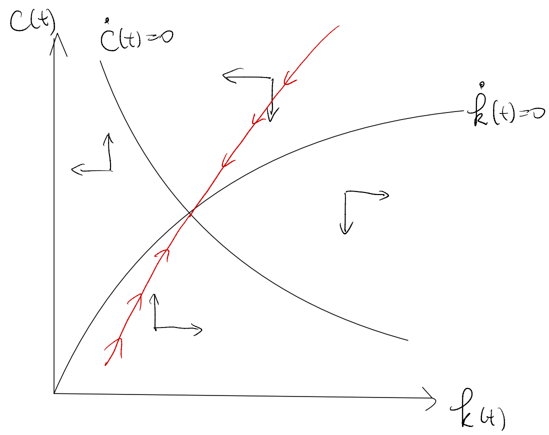

Figure 4: phase diagram

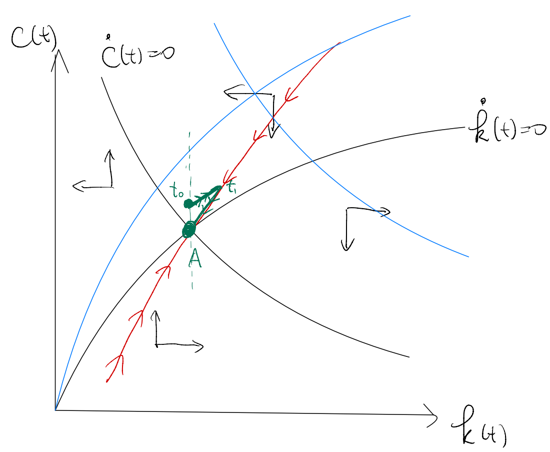

Figure 5: phase diagram

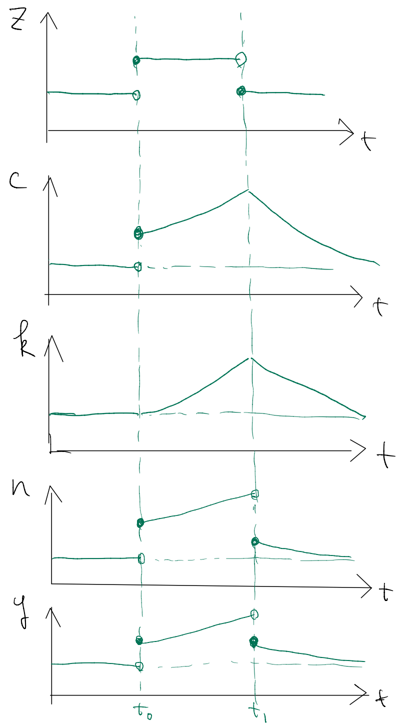

Figure 6: time series of variables

Figure 5 describes what happens when $z(t)$ goes up temporarily, as in the previous section. The $\dot{c}(t) = 0$ curve and $\dot{k}(t) = 0$ curve shift to the blue ones, in the same direction as in the previous section.

In this example, when $z(t)$ goes up by 1% (permanently), the new steady state (the crossing point of two blue curves) features $k(t)$ and $c(t)$ both increasing by 1.5%. $n(t)$ stays the same in the new steady state, reflecting the balanced-growth preferences. With 1% increase in $z(t)$, $c(t)$ has to go up by 3% with given $k(t)$ for $\dot{c}(t)$ to be equal to zero. For the second equation, a 1% increase in $z(t)$ is balanced by a 1% increase in $c(t)$ for given $k(t)$ to keep $\dot{k}(t)$ to be equal to zero. These magnitudes are reflected in the shifts of blue curves in the figure.

As we can see from the phase diagram, the jump of $c(t)$ with the temporary shock is less than 1%. The subsequent dynamics of $c(t)$ and $k(t)$ are similar to the previous section.

The dynamics of $n(t)$ involves some ambiguity. It jumps at t_0; because $z(t)$ jumps up more than $c(t)$ does, $n(t)$ jumps up at the impact, but the magnitude is less than 3/4 of the jump in $z(t)$. Because $c(t)$ and $k(t)$ both move up gradually between t_0 and t_1, $n(t)$ can go up or down during this period. At time t_1, $n(t)$ jumps down (reflecting the jump of $z(t)$, this time exactly 3/4 of $z(t)$ jump), but it can jump to above or below the original level. Then

$n(t)$ gradually converges to the original level.

The dynamics of $y(t)$ inherit the jumps of $z(t)$ and $n(t)$ initially, and then the movement of $y(t)$ is dictated by the combinations of $k(t)$ and $n(t)$ dynamics between t_0 and t_1. At time t_1, $y(t)$ jumps down, reflecting the jumps of $z(t)$ and $n(t)$.

In this section, $n(t)$ is added as an extra moving part, but its main role is to magnify the effect of $z(t)$ by jumping up and down in the same direction as $z(t)$. It may may have movements between t_0 and t_1 and after t_1, but it does not necessarily add a powerful channel of propagation. Overall, the dynamics of $y(t)$ is dictated by the dynamics of $z(t)$ (and $n(t)$, which has a magnifying role), and $k(t)$ provides a slow-moving propagation force. Given that $k(t)$ moves slowly, the dynamics of $y(t)$ over the business cycle will look quite similar to the dynamics of $z(t)$.