An Analytical Solution for the Solow Model

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This note presents the analytical solution for a special case in the continuous-time Solow model, based on Sørensen and Whitta-Jacobsen (2010). This is an exact solution, and makes it clear that the economy goes closer and closer to the steady state, but it never reaches it in a finite time.

1 Setup

There is a representative firm producing with the constant-returns-to-scale Cobb-Douglas technology

$$Y(t) = K(t)^{\alpha}(A(t)L(t))^{1-\alpha}, \tag{1}$$

where $Y(t)$ is the aggregate output, $K(t)$ is capital stock, $A(t)$ is the labor productivity (technology), and $L(t)$ is labor (population). $\alpha \in (0, 1)$.

The capital stock in the economy evolves following the differential equation

$$\dot{K}(t) = I(t) - \delta K(t), \tag{2}$$

where $I(t)$ is investment and $\delta > 0$ is the depreciation rate of capital. $\dot{K}(t) \equiv dK(t)/dt$ is the time derivative.

The consumers save a fraction $s \in (0, 1)$ of their income:

$$S(t) = sY(t), \tag{3}$$

where, in the goods market equilibrium, saving $S(t)$ equals investment.

$$S(t) = I(t). \tag{4}$$

Using (1), (3), and (4), the equation (2) can be rewritten as

$$\dot{K}(t) = sK(t)^{\alpha}(A(t)L(t))^{1-\alpha} - \delta K(t). \tag{5}$$

The population $L(t)$ is assumed to grow at the rate n and the technology $A(t)$ grows at the rate x:

$$\frac{\dot{L}(t)}{L(t)} = n \tag{6}$$

and

$$\frac{\dot{A}(t)}{A(t)} = x. \tag{7}$$

Let us define the variable $k(t)$ by

$$k(t) \equiv \frac{K(t)}{A(t)L(t)}. \tag{8}$$

This definition implies

$$\frac{\dot{k}(t)}{k(t)} = \frac{\dot{K}(t)}{K(t)} - \frac{\dot{A}(t)}{A(t)} - \frac{\dot{L}(t)}{L(t)}.$$

Using (5), (6), and (7), this equation can be rewritten as:

$$\frac{\dot{k}(t)}{k(t)} = \frac{sK(t)^\alpha (A(t)L(t))^{1-\alpha} - \delta K(t)}{K(t)} - x - n.$$

Rearranging (and using the definition (8), this equation is transformed to

$$\dot{k}(t) = sk(t)^\alpha - (n + x + \delta)k(t).$$

This is a differential equation with variable $k(t)$ only. We will work with this fundamental Solow equation.

2 Analytical solution

Repeating the fundamental Solow equation,

$$\dot{k}(t) = sk(t)^\alpha - (n + x + \delta)k(t). \tag{9}$$

First, note that the steady-state (balanced-growth) value of $k(t)$, k^*, can be solved analytically from

$$0 = s(k^*)^\alpha - (n + x + \delta)k^*.$$

The solution is

$$k^* = \left(\frac{s}{n + x + \delta}\right)^{\frac{1}{1-\alpha}}. \tag{10}$$

Define a variable $z(t)$ as

$$z(t) \equiv k(t)^{1-\alpha}. \tag{11}$$

In terms of the growth rates,

$$\frac{\dot{z}(t)}{z(t)} = (1 - \alpha)\frac{\dot{k}(t)}{k(t)}.$$

Plugging (9) into the right-hand side,

$$\frac{\dot{z}(t)}{z(t)} = (1 - \alpha)[sk(t)^{\alpha-1} - (n + x + \delta)].$$

Noting that $k(t)^{\alpha-1} = 1/z(t)$, this equation implies

$$\dot{z}(t) = (1-\alpha)s - (1-\alpha)(n+x+\delta)z(t). \tag{12}$$

This is a linear differential equation that can be solved easily. Let

$$\lambda \equiv (1-\alpha)(n+x+\delta).$$

Guess that the solution takes the form

$$z(t) = \mathcal{A} + \mathcal{B}e^{-\lambda t}. \tag{13}$$

To determine $\mathcal{A}$ and $\mathcal{B}$, first, denoting the initial value of $z(t)$ as z_0,

$$z_0 = \mathcal{A} + \mathcal{B}$$

has to hold. Second, because we know that $z(t) \to z^*$ as $t \to \infty$, where z^* is the steady-state value of $z(t)$ (from (10), $z^* = s/(n+x+\delta)$),

$$z^* = \mathcal{A}$$

has to hold.[1] Therefore, (13) has to be rewritten as

$$z(t) = z^* + (z_0 - z^*)e^{-\lambda t}.$$

One can then verify that this solution indeed satisfies the differential equation (12).

Therefore, from the definition of $z(t)$ in (11), the analytical solution of (9) is

$$k(t) = \left\{(k^*)^{1-\alpha} + \left[k_0^{1-\alpha} - (k^*)^{1-\alpha}\right] e^{-\lambda t}\right\}^{\frac{1}{1-\alpha}},$$

where k_0 is the initial value of $k(t)$ and k^* is defined in (10). It can be seen that when $k_0 \neq k^*$, the term $[k_0^{1-\alpha} - (k^*)^{1-\alpha}]e^{-\lambda t}$ becomes closer and closer to zero as time passes by, but never reaches zero in a finite time.

References

Sørensen, P. B. and H. J. Whitta-Jacobsen (2010). *Introducing Advanced Macroeconomics: Growth and Business Cycles* (2 ed.). McGraw-Hill.

[1] The second condition can alternatively be obtained by plugging (13) into (12) and compare coefficients.