Weitzman (1976)

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This note goes over Weitzman's (1976) basic result. Consider an economy with a consumption good and a capital good. Both goods are produced by capital. Consider the net national product (NNP) $Y(t)$ be defined as:

$$Y(t) \equiv C(t) + p(t)I(t), \tag{1}$$

where $Y(t)$ is output (NNP), $C(t)$ is (aggregate) consumption, $p(t)$ is the relative price of investment good, and $I(t)$ is net investment. Let $K(t)$ be the capital stock. Then

$$I(t) = \frac{dK(t)}{dt}.$$

Let the (representative) consumer maximize

$$\mathbf{U} \equiv \int_0^\infty e^{-\rho t} C(t) dt,$$

where $\rho > 0$ is the discount rate.

The result we will show is that, in the steady-state of the competitive equilibrium,

$$Y(t) = \rho \int_t^\infty e^{-\rho(s-t)} C(s) ds.$$

That is, the NNP represents the (average) present value if consumption. The consumer's budget constraint is

$$\dot{K}(t) = I(t)$$

and

$$Y(K(t)) = C(t) + p(t)I(t), \tag{2}$$

where $\dot{K}(t)$ is a shorthand notation of $dK(t)/dt$ and $Y(K(t))$ is the income (capital rental plus the firm ownership, adding up to the entire output). Note that the consumer is a price taker and when calculating how income changes with $K(t)$, the consumer takes $p(t)$ as given. That is,

$$Y'(K(t)) = \frac{dC(t)}{dK(t)} + p(t)\frac{dI(t)}{dK(t)}. \tag{3}$$

Setting up the Hamiltonian

$$H(t) \equiv e^{-\rho t} C(t) + \mu(t) I(t) + \lambda(t)(Y(K(t)) - C(t) - p(t)I(t)),$$

the first-order conditions are

$$e^{-\rho t} - \lambda(t) = 0,$$

$$\mu(t) - \lambda(t)p(t) = 0,$$

and

$$\lambda(t)Y'(K(t)) + \dot{\mu}(t) = 0.$$

Rearranging, we obtain

$$\dot{p}(t) = \rho p(t) - Y'(K(t)). \tag{4}$$

(This, of course, can also be interpreted as an asset arbitrage equation $\rho p(t) = Y'(K(t)) + \dot{p}(t)$.)

On the production side, suppose that the consumption good can be produced by technology

$$C(t) = F(K_C(t))$$

and the capital good can be produced by

$$I(t) = G(K_I(t)).$$

Here, $K_C(t)$ is the capital stock used for consumption good production and $K_I(t)$ is the capital stock used for capital good production. From the capital market clearing,

$$K_C(t) + K_I(t) = K(t) \tag{5}$$

holds. The consumption good producer maximizes

$$C(t) - r(t)K_C(t) = F(K_C(t)) - r(t)K_C(t),$$

where $r(t)$ is the rental rate of capital. The capital good producer maximizes

$$p(t)I(t) - r(t)K_I(t) = p(t)G(K_I(t)) - r(t)K_I(t).$$

The FOCs for both problems imply

$$F'(K_C(t)) = p(t)G'(K_I(t)). \tag{6}$$

Now, differentiate (1) with respect to t:

$$\frac{dY(t)}{dt} = \frac{dC(t)}{dt} + p(t)\frac{dI(t)}{dt} + \frac{dp(t)}{dt}I(t). \tag{7}$$

Because

$$\frac{dC(t)}{dt} = F'(K_C(t))\frac{dK_C(t)}{dt},$$

$$p(t)\frac{dI(t)}{dt} = p(t)G'(K_I(t))\frac{dK_I(t)}{dt},$$

and (6), together with the fact that (5) implies

$$\frac{dK_C(t)}{dt} + \frac{dK_I(t)}{dt} = \frac{dK(t)}{dt},$$

we obtain

$$\frac{dC(t)}{dt} + p(t)\frac{dI(t)}{dt} = F'(K_C(t))\frac{dK(t)}{dt} = F'(K_C(t))I(t). \tag{8}$$

From (4),

$$\frac{dp(t)}{dt}I(t) = [\rho p(t) - Y'(K(t))]I(t). \tag{9}$$

Note that from (3),

$$Y'(K(t)) = \frac{dC(t)}{dK(t)} + p(t)\frac{dI(t)}{dK(t)} = F'(K_C(t))\frac{dK_C(t)}{dK(t)} + p(t)G'(K_I(t))\frac{dK_I(t)}{dK(t)}.$$

From (5),

$$\frac{dK_C(t)}{dK(t)} + \frac{dK_I(t)}{dK(t)} = 1,$$

and thus combining with (6) we obtain

$$Y'(K(t)) = F'(K_C(t)).$$

Using this relationship, (8), and (9), (7) can be rewritten as

$$\frac{dY(t)}{dt} = \rho p(t)I(t) = \rho(Y(t) - C(t)).$$

Solving this differential equation, assuming $\lim_{s\to\infty} e^{-\rho s}Y(s) = 0$, we obtain

$$Y(t) = \rho \int_t^\infty e^{-\rho(s-t)}C(s)ds$$

as desired. The left-hand side (the current NNP) is equal to the (time-average of) the consumer's present value welfare.

To see some intuition, suppose that all future $C(s)$ is constant at $C(t)$. Then $Y(t) = C(t)$, or $I(t) = 0$. When $I(t) = 0$, the capital stock does not increase or decrease over time, and thus $Y(t)$ is constant over time. The optimal consumption is a constant value at $C(t)$. If $C(s)$ grows over time, $\rho \int_t^\infty e^{-\rho(s-t)}C(s)ds > C(t)$ and therefore $Y(t) > C(t)$, implying $I(t) > 0$. For consumption to grow over time, $Y(t)$ has to increase over time, and to achieve the growth, $I(t)$ has to be positive. Thus, the intuition is that, the NNP $(C(t) + p(t)I(t))$ measures the present value of welfare because the $C(t)$ part measures the consumption level and the $p(t)I(t)$ part measures the future growth potential of consumption.

References

Weitzman, M. L. (1976). On the Welfare Significance of National Product in a Dynamic Economy. *Quarterly Journal of Economics 90*, 156–162.