

Technical Appendix to “Asset Prices in a Huggett Economy”

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Section A of this Technical Appendix considers assets in positive net supply and Section B gathers the proofs of all propositions except those of Propositions 10 and 11, whose proofs can be found in the Appendix to the published article.

A Assets in positive net supply

We consider assets in positive supply here. The purpose is to show that such economies, both in the case with and without aggregate uncertainty, there is an equivalent economy with assets in zero supply but with appropriately adjusted, looser borrowing constraints. We demonstrate this in a slightly differently way than in the paper (where only the case without aggregate shocks is discussed). In particular, we show that if the economies with positive asset supplies are amended with the appropriate higher, and positive, values for the lower bound of assets—the risk-free asset in the first economy and the contingent claims in the second—then those economies have identical prices to those studied in the paper.

A.1 No aggregate uncertainty

Suppose that there is an asset, called a “tree,” that generates a constant amount η every period.¹⁸ Let the price of the tree be p and the individual holding of the tree be x . Then the individual consumer’s problem becomes

$$V_s(a, x) = \max_{c, a', x'} \frac{c^{1-\sigma}}{1-\sigma} + \beta[\pi_{sh}V_h(a', x') + (1 - \pi_{sh})V_\ell(a', x')]$$

subject to

$$c = a + (p + \eta)x + \epsilon_s - qa' - px'.$$

¹⁸A similar argument can be made if there is a constant positive supply of outside, say government, bonds, with an associated government budget constraint.

In equilibrium, the bond and the tree have to generate the same return (no arbitrage), so $(p + \eta)/p = 1/q$. Therefore, $p = q(p + \eta)$ holds. Using this, the budget constraint can be rewritten as

$$c = (a + (p + \eta)x) + \epsilon_s - q(a' + (p + \eta)x').$$

Let $\hat{a} \equiv a + (p + \eta)x$. Then the problem can be rewritten as

$$\hat{V}_s(\hat{a}) = \max_{c, \hat{a}'} \frac{c^{1-\sigma}}{1-\sigma} + \beta[\pi_{sh}\hat{V}_h(\hat{a}') + (1 - \pi_{sh})\hat{V}_\ell(\hat{a}')]]$$

subject to

$$c = \hat{a} + \epsilon_s - q\hat{a}'.$$

Now, suppose that the borrowing constraint is $\hat{a} \geq p + \eta$, i.e., we use a borrowing constraint on total wealth rather than on the holdings of individual assets. Below, we will show that the equilibrium is that $\hat{a} = p + \eta$ for everyone. One allocation that achieves this is $a = 0$ and $x = 1$ for everyone—that is, no one holds bonds and everyone owns the same amount of the tree. Other asset holding patterns are also possible—some can hold $a < 0$ and $x > 1$ while others can have $a > 0$ and $x < 1$. The only requirements for an equilibrium are that $\hat{a} = p + \eta$ for everyone, a sums up to zero, and x sums up to one.

To show that $\hat{a} = p + \eta$ for everyone is the only equilibrium, define $\tilde{a} \equiv \hat{a} - p - \eta$ and $\tilde{\epsilon}_s \equiv \epsilon_s + \eta$.¹⁹ Then the problem becomes

$$\tilde{V}_s(\tilde{a}) = \max_{c, \tilde{a}'} \frac{c^{1-\sigma}}{1-\sigma} + \beta[\pi_{sh}\tilde{V}_h(\tilde{a}') + (1 - \pi_{sh})\tilde{V}_\ell(\tilde{a}')]]$$

subject to

$$c = \tilde{a} + \tilde{\epsilon}_s - q\tilde{a}'$$

and

$$\tilde{a} \geq 0,$$

which is identical to the original problem. Therefore, the equilibrium is autarky: $\tilde{a} = 0$ and $c = \tilde{\epsilon}_s$. $\tilde{a} = 0$ implies $\hat{a} = p + \eta$. As long as the borrowing constraint is set at an

¹⁹This is the transformation that we use in the main text of the paper.

appropriate level, we can transform an economy where there are assets in positive net supply into an economy with a bond in zero net supply. Thus, the borrowing constraint here means that agents have to have at least a certain (positive) amount of the asset.

A.2 Aggregate uncertainty

First, we consider a case where there is one “tree” in addition to the two “Arrow securities.” Let the tree price at state z be p_z , the dividend of the tree at state z be η_z , and the tree holding be x . The problem becomes

$$V(a, x; s, z) = \max_{c, a'_g, a'_b, x'} \frac{c^{1-\sigma}}{1-\sigma} + \beta \left[\sum_{z'=g,b} \phi_{zz'} [\pi_{sh|zz'} V(a'_{z'}, x'; h, z') + (1 - \pi_{sh|zz'}) V(a'_{z'}, x'; \ell, z')] \right]$$

subject to

$$c = a + (p_z + \eta_z)x + \epsilon_s - Q_{zg}a'_g - Q_{zb}a'_b - p_z x'$$

and borrowing constraints. From arbitrage, $p_z = Q_{zg}(p_g + \eta_g) + Q_{zb}(p_b + \eta_b)$ has to hold. Thus the budget constraint can be rewritten as

$$c = (a + (p_z + \eta_z)x) + \epsilon_s - Q_{zg}(a'_g + (p_g + \eta_g)x') - Q_{zb}(a'_b + (p_b + \eta_b)x').$$

Let $\hat{a}_z = a_z + (p_z + \eta_z)x$. Then the problem can be rewritten as

$$\hat{V}(\hat{a}; s, z) = \max_{c, \hat{a}'_g, \hat{a}'_b} \frac{c^{1-\sigma}}{1-\sigma} + \beta \left[\sum_{z'=g,b} \phi_{zz'} [\pi_{sh|zz'} \hat{V}(\hat{a}'_{z'}; h, z') + (1 - \pi_{sh|zz'}) \hat{V}(\hat{a}'_{z'}; \ell, z')] \right]$$

subject to

$$c = \hat{a} + \epsilon_s - Q_{zg}\hat{a}'_g - Q_{zb}\hat{a}'_b.$$

Let us impose the borrowing constraints $\hat{a}'_g \geq p_g + \eta_g$ and $\hat{a}'_b \geq p_b + \eta_b$. We will show that in equilibrium $\hat{a}'_g = p_g + \eta_g$ and $\hat{a}'_b = p_b + \eta_b$. One set of asset holdings that can achieve this equilibrium is $a'_g = 0$, $a'_b = 0$, and $x' = 1$ for everyone. Again, it is important that the constraints are on the total amount of asset, rather than individual assets, and for example $a'_g < 0$, $a'_b < 0$ and $x' > 1$ for one consumer can be consistent with an equilibrium, as long as $\hat{a}'_g = p_g + \eta_g$ and $\hat{a}'_b = p_b + \eta_b$ are satisfied and the total asset demand equals the total supply for each asset.

Let $\tilde{a}_z \equiv \hat{a}_z - p_z - \eta_z$ and $\tilde{\epsilon}_{sz} \equiv \epsilon_s + \eta_z$. Then the problem can be rewritten as

$$\tilde{V}(\tilde{a}; s, z) = \max_{c, \tilde{a}'_g, \tilde{a}'_b} \frac{c^{1-\sigma}}{1-\sigma} + \beta \left[\sum_{z'=g,b} \phi_{zz'} [\pi_{sh|zz'} \tilde{V}(\tilde{a}'_{z'}; h, z') + (1 - \pi_{sh|zz'}) \tilde{V}(\tilde{a}'_{z'}; \ell, z')] \right]$$

subject to

$$c = \tilde{a} + \tilde{\epsilon}_{sz} - Q_{zg} \tilde{a}'_g - Q_{zb} \tilde{a}'_b,$$

with

$$\tilde{a}'_g \geq 0 \text{ and } \tilde{a}'_b \geq 0.$$

This is equivalent to our baseline problem, and therefore Q_{zg} and Q_{zb} are the same, except that ϵ_s is adjusted to $\tilde{\epsilon}_{sz}$. The equilibrium is autarky and the individual consumption is equal to $\tilde{\epsilon}_{sz}$. $\tilde{a}'_g = 0$ and $\tilde{a}'_b = 0$ imply that $\hat{a}'_g = p_g + \eta_g$ and $\hat{a}'_b = p_b + \eta_b$.

A.2.1 A representation with a bond and a stock

In the previous section, the stock (claim for the “tree”) was a redundant asset in the sense that the two aggregate states are already spanned by the Arrow securities. This allowed us to price the stock with arbitrage. In this section, we consider an economy where there are only two assets, a bond and a stock. As in the previous section, the stock yields η_z every period. One unit of the bond provides one unit of consumption good regardless of the aggregate state. The total supply of stock is 1 unit and the bond is in zero net supply. Assume $p_g + \eta_g \neq p_b + \eta_b$. Denote the stock holding by x and the bond holding by y . Let p_z be the stock price at state z and q_z be the bond price at z .

We first show that we can replicate the payoffs of Arrow securities by combining the bond and the stock with appropriate proportions. Let us define

$$x'_g \equiv \frac{1}{p_g + \eta_g - p_b - \eta_b}$$

and

$$y'_g \equiv -\frac{p_b + \eta_b}{p_g + \eta_g - p_b - \eta_b}.$$

Then, we can easily see that

$$y'_g + (p_g + \eta_g)x'_g = 1$$

and

$$y'_g + (p_b + \eta_b)x'_g = 0$$

are satisfied. This means that by holding y'_g units of the bond and x'_g units of the stock, one can guarantee to receive 1 unit if the next period aggregate state is g and receive 0 unit if the next period aggregate state is b . Therefore, holding the bundle (x'_g units of stock, y'_g units of bond) is identical to a g -state Arrow security. Similarly, the bundle of

$$x'_b \equiv -\frac{1}{p_g + \eta_g - p_b - \eta_b}$$

units of the stock and

$$y'_b \equiv \frac{p_g + \eta_g}{p_g + \eta_g - p_b - \eta_b}$$

units of the bond yields an identical payoff to a b -state Arrow security. Let the cost of acquiring these bundles be Q_{zg} and Q_{zb} . That is,

$$Q_{zg} \equiv \frac{p_z}{p_g + \eta_g - p_b - \eta_b} - \frac{q_z(p_b + \eta_b)}{p_g + \eta_g - p_b - \eta_b}$$

and

$$Q_{zb} \equiv -\frac{p_z}{p_g + \eta_g - p_b - \eta_b} + \frac{q_z(p_g + \eta_g)}{p_g + \eta_g - p_b - \eta_b}.$$

We can easily check the following simple relationships between $Q_{zz'}$ and the stock/bond prices:

$$Q_{zg} + Q_{zb} = q_z \tag{24}$$

and

$$(p_g + \eta_g)Q_{zg} + (p_b + \eta_b)Q_{zb} = p_z. \tag{25}$$

Let $a'_{z'}$ be the demand of the z' -state security bundle. Then the corresponding total demand for stock is

$$x' = x'_g a'_g + x'_b a'_b = \frac{a'_g - a'_b}{p_g + \eta_g - p_b - \eta_b} \tag{26}$$

and the total demand for bonds is

$$y' = y'_g a'_g + y'_b a'_b = \frac{a'_b(p_g + \eta_g) - a'_g(p_b + \eta_b)}{p_g + \eta_g - p_b - \eta_b}. \tag{27}$$

Note that there is one-to-one correspondence between (x', y') and (a'_g, a'_b) . That is, by demanding two sets of bundles, the consumers are indirectly demanding the stock and the bond. By adjusting the bundle demands a'_g and a'_b , they can adjust the demands for stocks and bonds as if they were directly choosing x' and y' . Therefore, if (a'_g, a'_b) maximizes the utility given $Q_{zz'}$, the corresponding (x', y') from (26) and (27) also maximizes the utility given the prices (p_z, q_z) that satisfy (24) and (25). The budget constraint for the original bond-and-stock economy is

$$c = (p_z + \eta_z)x + y + \epsilon_s - p_z x' - q_z y'.$$

Using (24), (25), (26), and (27), this can be rewritten as

$$c = a_z + \epsilon_s - Q_{zg}a'_g - Q_{zb}a'_b.$$

Now, let us impose the borrowing constraint $a'_g \geq p_g + \eta_g$ and $a'_b \geq p_b + \eta_b$. Since $a'_{z'} = (p_{z'} + \eta_{z'})x' + y'$ from (26) and (27), these are equivalent to $(p_g + \eta_g)(x' - 1) + y' \geq 0$ and $(p_b + \eta_b)(x' - 1) + y' \geq 0$.

As in the previous section, consider the transformation

$$\tilde{a}_z = a_z - p_z - \eta_z \tag{28}$$

and

$$\tilde{\epsilon}_{sz} = \epsilon_s + \eta_z. \tag{29}$$

Then the budget constraint and the borrowing constraint can be rewritten as

$$c = \tilde{a}_z + \tilde{\epsilon}_{sz} - Q_{zg}\tilde{a}'_g - Q_{zb}\tilde{a}'_b.$$

and

$$\tilde{a}'_g \geq 0 \text{ and } \tilde{a}'_b \geq 0.$$

In sum, we have demonstrated the equivalence of the problem

$$V(x, y; s, z) = \max_{c, x', y'} \frac{c^{1-\sigma}}{1-\sigma} + \beta \left[\sum_{z'=g,b} \phi_{zz'} [\pi_{sh|zz'} V(x', y'; h, z') + (1 - \pi_{sh|zz'}) V(x', y'; \ell, z')] \right] \tag{P1}$$

subject to

$$c = (p_z + \eta_z)x + y + \epsilon_s - p_z x' - q_z y'$$

and

$$(p_g + \eta_g)(x' - 1) + y' \geq 0 \text{ and } (p_b + \eta_b)(x' - 1) + y' \geq 0;$$

and the problem

$$\tilde{V}(\tilde{a}; s, z) = \max_{c, \tilde{a}'_g, \tilde{a}'_b} \frac{c^{1-\sigma}}{1-\sigma} + \beta \left[\sum_{z'=g,b} \phi_{zz'} [\pi_{sh|zz'} \tilde{V}(\tilde{a}'_{z'}; h, z') + (1 - \pi_{sh|zz'}) \tilde{V}(\tilde{a}'_{z'}; \ell, z')] \right] \quad (P2)$$

subject to

$$c = \tilde{a} + \tilde{\epsilon}_{sz} - Q_{zg} \tilde{a}'_g - Q_{zb} \tilde{a}'_b$$

and

$$\tilde{a}'_g \geq 0 \text{ and } \tilde{a}'_b \geq 0,$$

where $\tilde{\epsilon}_{sz} = \epsilon_s + \eta_z$. The prices are one-to-one linked by (24) and (25), and the quantities are one-to-one linked by (26), (27), (28), and (29). The second problem (P2) is familiar to us: the equilibrium is autarky. It means that the borrowing constraints hold with equality, which in turn implies that the borrowing constraints in the first problem (P1) also hold with equality. Therefore, $x' = 1$ and $y' = 0$ hold in equilibrium (there is no indeterminacy as in the previous section because there is no redundant asset). Q_{zg} and Q_{zb} are determined in a familiar manner and this can be translated into p_z and q_z using (24) and (25).

The equivalence of the two problems can also be seen from the Euler equations. Recall that the Euler equation for (P2) with autarky is

$$-Q_{zz'} \tilde{\epsilon}_{sz}^{-\sigma} + \beta \phi_{zz'} [\pi_{sh|zz'} \tilde{\epsilon}_{hz'}^{-\sigma} + (1 - \pi_{sh|zg}) \tilde{\epsilon}_{\ell z'}^{-\sigma}] + \lambda_{sz}^{z'} = 0, \quad (30)$$

for $z' = g, b$. By adding this up for $z' = g$ and $z' = b$, we obtain

$$-q_z \tilde{\epsilon}_{sz}^{-\sigma} + \beta \left[\sum_{z'=g,b} (\phi_{zz'} [\pi_{sh|zz'} \tilde{\epsilon}_{hz'}^{-\sigma} + (1 - \pi_{sh|zg}) \tilde{\epsilon}_{\ell z'}^{-\sigma}] + \lambda_{sz}^{z'}) \right] = 0, \quad (31)$$

where we used the relationship (24). By multiplying $(p_{z'} + \eta_{z'})$ on each sides of (30) and add up for $z' = g$ and $z' = b$, we obtain

$$-p_z \tilde{\epsilon}_{sz}^{-\sigma} + \beta \left[\sum_{z'=g,b} (p_{z'} + \eta_{z'}) (\phi_{zz'} [\pi_{sh|zz'} \tilde{\epsilon}_{hz'}^{-\sigma} + (1 - \pi_{sh|zg}) \tilde{\epsilon}_{lz'}^{-\sigma}] + \lambda_{sz}') \right] = 0, \quad (32)$$

where we used the relationship (25). It is straightforward to see that (31) and (32) are the Euler equations for (P1) with $x' = 1$ and $y' = 0$. Therefore, if (30) holds (that is, $\tilde{a}'_g = 0$ and $\tilde{a}'_b = 0$ are the optimal choices in (P2) given $Q_{zz'}$), (31) and (32) also hold and $x' = 1$ and $y' = 0$ are the optimal choices given q_z and p_z in (P1).

B Proofs of Propositions

Proof of Proposition 2: From (16) and $E[m_{zz'}] = q_z > 0$, $E[R_{zz'}^e]$ is positive if and only if $Cov(R_{zz'}^e, m_{zz'})$ is negative. From the definitions of $R_{zz'}^e$, $R_{zz'}$, and $m_{zz'}$, $Cov(R_{zz'}^e, m_{zz'})$ is negative if and only if $Cov(Y_{z'}, \pi_{hh|zz'})$ is positive. \square

Proof of Proposition 3: From the definitions of the risk-free rate and the expected return on the risky asset, the multiplicative risk premium is

$$\frac{E[R_{zz'}^e]}{R_z^f} = \frac{(\sum_{z'=g,b} Q_{zz'}) (\sum_{z'=g,b} \phi_{zz'} Y_{z'})}{\sum_{z'=g,b} Q_{zz'} Y_{z'}}.$$

Denote the Arrow security price in the complete-markets economy by $Q_{zz'}^c$ and the Arrow security price in the incomplete-markets economy as $Q_{zz'}^i$. Clearly, for $E[R_{zz'}^e]/R_z^f$ to be the same for both economies (and for any $Y_{z'}$), there has to exist a number $\theta_z > 0$ that is independent of z' and satisfy $Q_{zz'}^i = \theta_z Q_{zz'}^c$ (and if this is the case, then θ_z cancels out and the equivalence holds). Thus, a necessary and sufficient condition for irrelevance is

$$\frac{Q_{zg}^i}{Q_{zb}^i} = \frac{Q_{zg}^c}{Q_{zb}^c} \quad (33)$$

for $z = g, b$. The Arrow security prices are determined by

$$Q_{zz'}^i = \beta \phi_{zz'} \left[\pi_{hh} \left(\frac{\epsilon_{hz'}}{\epsilon_{hz}} \right)^{-\sigma} + \pi_{hl} \left(\frac{\epsilon_{lz'}}{\epsilon_{hz}} \right)^{-\sigma} \right]$$

and

$$Q_{zz'}^c = \beta \phi_{zz'} \left(\frac{C_{z'}}{C_z} \right)^{-\sigma},$$

where aggregate consumption C_z is

$$C_z = \chi_h \epsilon_{hz} + \chi_\ell \epsilon_{\ell z} = \frac{\pi_{\ell h} \epsilon_{hz} + (1 - \pi_{hh}) \epsilon_{\ell z}}{1 - \pi_{hh} + \pi_{\ell h}},$$

where the second equality uses (19). Therefore, (33) becomes

$$\frac{\pi_{hh} \left(\frac{\epsilon_{hg}}{\epsilon_{hz}} \right)^{-\sigma} + \pi_{h\ell} \left(\frac{\epsilon_{\ell g}}{\epsilon_{hz}} \right)^{-\sigma}}{\pi_{hh} \left(\frac{\epsilon_{hb}}{\epsilon_{hz}} \right)^{-\sigma} + \pi_{h\ell} \left(\frac{\epsilon_{\ell b}}{\epsilon_{hz}} \right)^{-\sigma}} = \frac{\left(\frac{\pi_{\ell h} \epsilon_{hg} + (1 - \pi_{hh}) \epsilon_{\ell g}}{\pi_{\ell h} \epsilon_{hz} + (1 - \pi_{hh}) \epsilon_{\ell z}} \right)^{-\sigma}}{\left(\frac{\pi_{\ell h} \epsilon_{hb} + (1 - \pi_{hh}) \epsilon_{\ell b}}{\pi_{\ell h} \epsilon_{hz} + (1 - \pi_{hh}) \epsilon_{\ell z}} \right)^{-\sigma}},$$

which is equivalent to (20). Note that (20) does not depend on z (all the terms that depend on z cancel out). \square

Proof of Proposition 4: From (9), for any $m_{zz'} \in [\beta, \beta\omega]$, we can find a $\pi_{hh|zz'} \in [0, 1]$ value that generates this value of $m_{zz'}$. The upper bound can be made arbitrarily large by making ϵ_h/ϵ_ℓ large. \square

Proof of Proposition 5: The first-order conditions in the case of varying ϵ_h and ϵ_ℓ are (denoting $z = z_t$ and $z' = z_{t+1}$)

$$\frac{Q_{t+1}(z^{t+1})}{\beta \phi_{zz'}} - \frac{\lambda_{h,t+1}(z^{t+1})}{\beta \phi_{zz'} \epsilon_{ht}(z^t)^{-\sigma}} = \pi_{t+1}(h|h, z^{t+1}) \left(\frac{\epsilon_{h,t+1}(z^{t+1})}{\epsilon_{ht}(z^t)} \right)^{-\sigma} + (1 - \pi_{t+1}(h|h, z^{t+1})) \left(\frac{\epsilon_{\ell,t+1}(z^{t+1})}{\epsilon_{ht}(z^t)} \right)^{-\sigma} \quad (34)$$

and

$$\frac{Q_{t+1}(z^{t+1})}{\beta \phi_{zz'}} - \frac{\lambda_{\ell,t+1}(z^{t+1})}{\beta \phi_{zz'} \epsilon_{\ell t}(z^t)^{-\sigma}} = \pi_{t+1}(h|\ell, z^{t+1}) \left(\frac{\epsilon_{h,t+1}(z^{t+1})}{\epsilon_{\ell t}(z^t)} \right)^{-\sigma} + (1 - \pi_{t+1}(h|\ell, z^{t+1})) \left(\frac{\epsilon_{\ell,t+1}(z^{t+1})}{\epsilon_{\ell t}(z^t)} \right)^{-\sigma}, \quad (35)$$

where $Q_{t+1}(z^{t+1})$ is the Arrow-security price and $\lambda_{s_t,t+1}(z^{t+1})$ is the Lagrange multiplier.

In the following, we will construct $\pi_{t+1}(s_{t+1}|s_t, z^{t+1})$, $\epsilon_{ht}(z^t)$, and $\epsilon_{\ell t}(z^t)$ that deliver a given $m_{t+1}(z^{t+1})$. Consider the individual income levels

$$\epsilon_{\ell t}(z^t) = 2\zeta C_t(z^t) \quad (36)$$

and

$$\epsilon_{ht}(z^t) = 2(1 - \zeta)C_t(z^t), \quad (37)$$

where $\zeta \in (0, 1)$. Later we will impose $\zeta < 1/2$ so that $\epsilon_{ht}(z^t) > \epsilon_{\ell t}(z^t)$.

Suppose that the initial population of the consumers who have the initial endowment is $1/2$ for both ℓ and h . Further suppose that the idiosyncratic probabilities are such that the population of each endowment consumers remain as $1/2$ forever. (We will explicitly spell out this condition later.)

First, note that the individual endowment (36) and (37) are consistent with the definition of the aggregate endowment:

$$C_t(z^t) \equiv \frac{\epsilon_{\ell t}(z^t)}{2} + \frac{\epsilon_{ht}(z^t)}{2} = \zeta C_t(z^t) + (1 - \zeta)C_t(z^t).$$

We will select $\pi_{t+1}(h|\ell, z^{t+1})$ so that the each endowment population is constant over time. This implies that

$$\frac{1}{2}(1 - \pi_{t+1}(h|h, z^{t+1})) = \frac{1}{2}\pi_{t+1}(h|\ell, z^{t+1}).$$

Therefore,

$$\pi_{t+1}(h|\ell, z^{t+1}) = 1 - \pi_{t+1}(h|h, z^{t+1}).$$

Thus, we automatically obtain $\pi_{t+1}(h|\ell, z^{t+1})$ from this equation once $\pi_{t+1}(h|h, z^{t+1})$ is assigned. Note that $\pi_{t+1}(h|\ell, z^{t+1}) \in [0, 1]$ is ensured if $\pi_{t+1}(h|h, z^{t+1}) \in [0, 1]$ is satisfied.

Inserting the income levels (36) and (37) into (34) and (35), we obtain

$$\left(\frac{Q_{t+1}(z^{t+1})}{\beta\phi_{zz'}} - \frac{\lambda_{h,t+1}(z^{t+1})}{\beta\phi_{zz'}\epsilon_{ht}(z^t)^{-\sigma}} \right) \left(\frac{C_{t+1}(z^{t+1})}{C_t(z^t)} \right)^\sigma = \pi_{t+1}(h|h, z^{t+1}) + (1 - \pi_{t+1}(h|h, z^{t+1})) \left(\frac{\zeta}{1 - \zeta} \right)^{-\sigma}$$

and

$$\left(\frac{Q_{t+1}(z^{t+1})}{\beta\phi_{zz'}} - \frac{\lambda_{\ell,t+1}(z^{t+1})}{\beta\phi_{zz'}\epsilon_{\ell t}(z^t)^{-\sigma}} \right) \left(\frac{C_{t+1}(z^{t+1})}{C_t(z^t)} \right)^\sigma = \pi_{t+1}(h|\ell, z^{t+1}) \left(\frac{1 - \zeta}{\zeta} \right)^{-\sigma} + (1 - \pi_{t+1}(h|\ell, z^{t+1})).$$

Therefore, $\lambda_{\ell,t+1}(z^{t+1}) > 0$ holds, and the low-endowment consumers are always borrowing constrained. This means that the high-endowment consumers' marginal rates

of substitution determine the pricing kernel. The pricing kernel $m_{t+1}(z^{t+1})$ is

$$m_{t+1}(z^{t+1}) = \beta \left(\frac{C_{t+1}(z^{t+1})}{C_t(z^t)} \right)^{-\sigma} \left[\pi_{t+1}(h|h, z^{t+1}) + (1 - \pi_{t+1}(h|h, z^{t+1})) \left(\frac{\zeta}{1 - \zeta} \right)^{-\sigma} \right]. \quad (38)$$

Thus, any

$$m_{t+1}(z^{t+1}) \in \left[\beta \left(\frac{C_{t+1}(z^{t+1})}{C_t(z^t)} \right)^{-\sigma}, \beta \left(\frac{C_{t+1}(z^{t+1})}{C_t(z^t)} \right)^{-\sigma} \left(\frac{\zeta}{1 - \zeta} \right)^{-\sigma} \right]$$

can be chosen by picking $\pi_{t+1}(h|h, z^{t+1}) \in [0, 1]$ appropriately, for a given ζ . Note that we do not have any restriction on ζ at this point, other than $0 < \zeta < 1/2$.

Pick ζ small enough so that

$$\left(\frac{\zeta}{1 - \zeta} \right)^{-\sigma} \geq \sup_{t, z^t, z^{t+1}} \frac{m_{t+1}(z^{t+1})}{\beta \left(\frac{C_{t+1}(z^{t+1})}{C_t(z^t)} \right)^{-\sigma}}$$

is satisfied. Then, all $m_{t+1}(z^{t+1}) \geq \beta \left(\frac{C_{t+1}(z^{t+1})}{C_t(z^t)} \right)^{-\sigma}$ at each z^{t+1} can be achieved by selecting $\pi_{t+1}(h|h, z^{t+1}) \in [0, 1]$ for each z^{t+1} to satisfy (38) for this ζ . \square

Proof of Proposition 7: M_i can be rewritten as $M_i = -\beta \sum_{j=1}^N \pi_{ij} (-u'(\epsilon_j)/u'(\epsilon_i))$. Clearly, $(-u'(\epsilon_j)/u'(\epsilon_i))$ is increasing in j . From the definition of first-order stochastic dominance (see, for example, Mas-Colell, Whinston, and Green (1995, Definition 6.D.1)),

$$\sum_{j=1}^N \hat{\pi}_{ij} \left(-\frac{u'(\epsilon_j)}{u'(\epsilon_i)} \right) > \sum_{j=1}^N \pi_{ij} \left(-\frac{u'(\epsilon_j)}{u'(\epsilon_i)} \right)$$

holds. Therefore, M_i is smaller for each $i = 2, \dots, N$ under $\hat{\pi}$ than under π . Since M_i is smaller for each $i = 2, \dots, N$, $\max_{i=2, \dots, N} M_i$ is also smaller. \square

Proof of Proposition 8: From the definition of M_i , it is sufficient to show that $H_i(\sigma) \equiv \sum_{j=1}^N \pi_{ij} (\epsilon_i/\epsilon_j)^\sigma$ is increasing in σ when $H_i(\sigma) \geq 1$ (since $H_N(\sigma) \geq 1$, it is always the case that $\max_{i=2, \dots, N} H_i(\sigma) \geq 1$). Note that $\sigma \geq 0$ and $H_i(0) = 1$. Let $\epsilon_i/\epsilon_j = k_{ij}$. Differentiating,

$$H'_i(\sigma) = \sum_{j=1}^N \pi_{ij} \log(k_{ij}) (k_{ij})^\sigma$$

and

$$H_i''(\sigma) = \sum_{j=1}^N \pi_{ij} (\log(k_{ij}))^2 (k_{ij})^\sigma.$$

Since $H_i''(\sigma) \geq 0$ and $H_i(0) = 1$, $H_i(\sigma)$ is always increasing for $\sigma \geq 0$ when $H_i(\sigma) \geq 1$.

□

Proof of Proposition 9: Equation (22) implies that for any $i = 2, \dots, N - 1$,

$$\sum_{j=1}^N \pi_{Nj} \frac{u'(\epsilon_j)}{u'(\epsilon_N)} \geq \sum_{j=1}^{i-1} \pi_{ij} \frac{u'(\epsilon_j)}{u'(\epsilon_N)} + \sum_{j=i}^N \pi_{ij}. \quad (39)$$

To see why this holds, note that the left-hand side can be rewritten as $\int_0^1 F(x)dx$, where

$$F(x) \equiv \begin{cases} \frac{u'(\epsilon_1)}{u'(\epsilon_N)} & \text{when } 0 \leq x \leq \pi_{N1}, \\ \frac{u'(\epsilon_j)}{u'(\epsilon_N)} & \text{when } \sum_{k=1}^{j-1} \pi_{Nk} < x \leq \sum_{k=1}^j \pi_{Nk}, \text{ where } 2 \leq j \leq N. \end{cases}$$

The right-hand side can be rewritten as $\int_0^1 G(x)dx$, where

$$G(x) \equiv \begin{cases} \frac{u'(\epsilon_1)}{u'(\epsilon_N)} & \text{when } 0 \leq x \leq \pi_{i1}, \\ \frac{u'(\epsilon_j)}{u'(\epsilon_N)} & \text{when } \sum_{k=1}^{j-1} \pi_{ik} < x \leq \sum_{k=1}^j \pi_{ik}, \text{ where } 2 \leq j \leq i-1, \\ 1 & \text{when } \sum_{k=1}^{j-1} \pi_{ik} < x \leq \sum_{k=1}^j \pi_{ik}, \text{ where } i \leq j \leq N. \end{cases}$$

Equation (22) ensures that $F(x) \geq G(x)$ for all x .

The following can be verified by comparing term by term:

$$\sum_{j=1}^{i-1} \pi_{ij} \frac{u'(\epsilon_j)}{u'(\epsilon_N)} + \sum_{j=i}^N \pi_{ij} \geq \sum_{j=1}^N \pi_{ij} \frac{u'(\epsilon_j)}{u'(\epsilon_i)}.$$

From this and (39), we obtain

$$\sum_{j=1}^N \pi_{Nj} \frac{u'(\epsilon_j)}{u'(\epsilon_N)} \geq \sum_{j=1}^N \pi_{ij} \frac{u'(\epsilon_j)}{u'(\epsilon_i)}$$

for any i , which is the desired inequality. □

References

- [1] Mas-Colell, A. M.D. Whinston, and J.R. Green (1995), *Microeconomic Theory*, New York: Oxford University Press.