Capital Accumulation by Firms

Toshihiko Mukoyama

May 2025

1 Fiction: consumer renting capital to firms

Consider the standard Ramsey growth model. There is a mass one of homogeneous, price-taking consumers. The representative consumer solves

$$\max_{\{c_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$c_t + k_{t+1} = r_t k_t + w_t \ell + (1-\delta) k_t,$$

where c_t is consumption at period t, k_t is capital stock holding, r_t is the rental rate of capital, w_t is wage rate, $\bar{\ell}$ is (fixed) individual labor supply.

There is a mass one of price-taking firms. Firms produce output using the constant-returns-to-scale production function $F(K_t^f, L_t^f)$, where K_t^f is capital input and L_t^f is labor input. The firms decide how much to demand K_t^f and L_t^f, maximizing the profit

$$F(K_t^f, L_t^f) - r_t K_t^f - w_t L_t^f.$$

In the market equilibrium, supply equals demand for both capital and labor. Therefore, $k_t = K_t^f = K_t$ and $\ell = L_t^f = L_t$ hold, where K_t and L_t are aggregate capital and labor in equilibrium.

The consumer's Euler equation can be written as

$$u'(c_t) = \beta(1 + r_{t+1} - \delta) u'(c_{t+1}).$$

In equilibrium where $c_t = C_t$, where C_t is the aggregate consumption, this equation can be rewritten as

$$u'(C_t) = \beta(1 + r_{t+1} - \delta) u'(C_{t+1}).$$

One of the first-order conditions for the firm is

$$F_1(K_t^f, L_t^f) = r_t,$$

where $F_i(\cdot,\cdot)$ denotes the partial derivative with respect to the ith term. In equilibrium, this equation can be rewritten as

$$F_1(K_t, \ell) = r_t.$$

Using this equation, we can rewrite the consumer's Euler equation as

$$u'(C_t) = \beta(1 + F_1(K_{t+1}, \ell) - \delta)u'(C_{t+1}). \tag{1}$$

Also, using the other first-order condition for the firm,

$$F_2(K_t, \ell) = w_t,$$

and the fact that $F_1(K_t, \ell)K_t + F_2(K_t, \ell)\ell = F(K_t, \ell)$ because $F(\cdot, \cdot)$ is constant returns, we can rewrite the consumer's budget constraint as

$$C_t + K_{t+1} = F(K_t, \ell) + (1 - \delta)K_t. \tag{2}$$

The two difference equations (1) and (2), combined with appropriate boundary conditions, determine the dynamic paths of C_t and K_t.

Of course, the setting so far appears fictional compared to modern reality. In the modern economy, a large part of the capital stock is owned by firms. We often use this fictional setting, nevertheless, because it is significantly simpler than the alternative settings where the firms own capital (and firms are owned by the consumers). In the following two sections, we show that the allocation will be the same in the alternative settings where firms own the capital stock.

2 Direct ownership

Suppose that the firms are directly owned by the consumers. Suppose that the firm produces with a "backyard production"—the owner works at her own firm. The firm owns (and invests in) the capital stock. Because the consumer is also the owner, she can determine the investment i_t and the capital stock. The consumer's problem is

$$\max_{\{c_t, i_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$c_t = \pi_t,$$

where π_t is the firm's profit

$$\pi_t = F(k_t, \ell) - i_t,$$

and

$$i_t = k_{t+1} - (1 - \delta)k_t.$$

Therefore, the consumer's budget constraint can be rewritten as

$$c_t = F(k_t, \ell) - (k_{t+1} - (1 - \delta)k_t). \tag{3}$$

The first-order condition yields the Euler equation, which in equilibrium can be written as

$$u'(C_t) = \beta(F_1(K_{t+1}, \ell) + 1 - \delta)u'(C_{t+1}).$$

This equation is identical to (1). It is also straightforward to see that the consumer's budget constraint (3) in equilibrium is identical to (2) (note that $k_t = K_t$ here because of the homogeneity of consumers).

3 Stock ownership

The previous section still looks somewhat distant from reality, given that modern companies are not operated through backyard production. In this section, suppose that the firms are owned through stock holdings, and they pay the profit as dividends to their owners. The consumer's problem is

$$\max_{\{c_t, x_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$c_t + (x_{t+1} - x_t)V_t = w_t \ell + x_t D_t, \tag{4}$$

where $x_t \in [0, 1]$ is the stock ownership (which sums up to one in the entire economy), and D_t is the total dividend from all firms in the economy. V_t is the value (stock price) of all firms in the economy.

The first-order condition for the consumer (in equilibrium, where $c_t = C_t$) yields the Euler equation

$$V_t u'(C_t) = \beta(D_{t+1} + V_{t+1})u'(C_{t+1}).$$

Using this equation for $t = 0, 1, ...$ and plugging in repeatedly, we obtain

$$V_0 u'(C_0) = \sum_{t=1}^{T} \beta^t u'(C_t) D_t + \beta^T u'(C_T) V_T.$$

Assuming that $\lim_{T\to\infty} \beta^T u'(C_T)V_T = 0$, we obtain

$$V_0 = \sum_{t=1}^{\infty} \beta^t \frac{u'(C_t)}{u'(C_0)} D_t,$$

which is the Lucas asset pricing formula.

Assume that the firm chooses labor demand and investment so that its value is maximized:

$$\max_{K_{t+1}^f, L_t^f, I_t^f} V_0 = \sum_{t=1}^{\infty} \beta^t \frac{u'(C_t)}{u'(C_0)} D_t, \tag{$P1$}$$

where

$$D_t = F(K_t^f, L_t^f) - w_t L_t^f - I_t^f \tag{5}$$

and

$$I_t^f = K_{t+1}^f - (1-\delta)K_t^f. \tag{6}$$

The first-order conditions for the firm yield

$$-\beta^t \frac{u'(C_t)}{u'(C_0)} + \beta^{t+1}\frac{u'(C_{t+1})}{u'(C_0)}(F_1(K_{t+1}^f, L_{t+1}^f) + (1-\delta)) = 0$$

and

$$F_2(K_t^f, L_t^f) = w_t.$$

The first equation, in equilibrium, can be rewritten as

$$u'(C_t) = \beta(F_1(K_{t+1}, \ell) + 1 - \delta)u'(C_{t+1}),$$

which is identical to (1). The equations (4), (5), (6) can be combined with the equilibrium conditions $c_t = C_t$, $k_t = K_t^f = K_t$, $\ell = L_t^f$, and $x_t = 1$ to obtain the identical equation to (2).

In summary, we have shown that the three settings, (i) consumers own and rent capital stock to firms every period, (ii) consumers own firms directly and accumulate capital, and (iii) consumers own firms through stock ownership and the firms accumulate capital so that they maximize their stock market value, yields the same equilibrium outcome. This result provides a justification for a large body of macroeconomic studies that use the first approach (despite it appearing unrealistic). The above derivation also clarifies why we need to use the stochastic discount factor $\beta^t u'(C_t)/u'(C_0)$ when considering the firm's dynamic decision problem ($P1$). Note that when there are frictions in the economy, such as taxes and financial constraints, these approaches may not yield the same results.