Chapter 3: The Solow Model

May 2026

Outline

Introduction

Motivation

Motivating fact: The capital-output ratio K_t/Y_t is roughly stable at ≈ 3 over time.

- Earlier theories (Harrod-Domar): treated this as a technological property (rigid, fixed proportions)
- But capital and labor are substitutable
- Why is K_t/Y_t approximately constant despite substitutability?

Solow's insight: Reconcile the stable capital-output ratio through **neoclassical** forces (decreasing marginal returns).

Two main takeaways:

1. The fundamental source of long-run per capita income growth is growth in technology A_t
2. If parameter values are common, different economies **converge** to the same income per capita in the long run

The Basic Model

The Aggregate Production Function

$$Y_t = F(K_t, L_t)$$

Assumptions on $F(K, L)$:

1. Strictly increasing in both K and L
2. Strictly quasiconcave (strictly convex isoquants)
3. Constant returns to scale: $F(cK, cL) = cF(K, L)$ for all $c > 0$
4. $F(0, L) = 0$
5. Inada conditions: $\lim_{K \to 0} F_1(K, L) = \infty$ and $\lim_{K \to \infty} F_1(K, L) = 0$

Normalize population and hours to 1: $L_t = 1$. Define $f(k_t) \equiv F(k_t, 1)$:

$$y_t = f(k_t)$$

Capital Accumulation and the Fundamental Equation

Capital accumulation:

$$k_{t+1} = i_t + (1-\delta)k_t$$

Goods market clearing: $y_t = c_t + i_t$

Saving behavior (exogenous, constant saving rate):

$$i_t = sy_t, \qquad s \in (0,1)$$

Substituting:

$$k_{t+1} = (1-\delta)k_t + sf(k_t)$$

This is the **fundamental equation of the Solow model**. Given k_0, the entire path $\{k_t, y_t, c_t, i_t\}$ is determined.

Steady State

A **steady state** is a constant $\bar{k}$ satisfying $k_t = \bar{k}$ for all t:

$$\bar{k} = (1-\delta)\bar{k} + sf(\bar{k}) \qquad \Rightarrow \qquad \delta\bar{k} = sf(\bar{k})$$

- LHS: straight line through origin with slope δ
- RHS: strictly increasing, strictly concave, slope ∞ at origin, slope $\to 0$
- $\Rightarrow$ Unique positive intersection (guaranteed by Inada conditions)

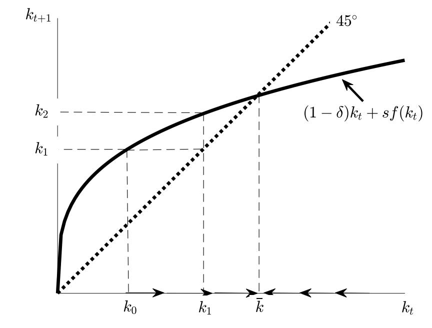
Cobb-Douglas: $f(k) = k^{\alpha}$, $\alpha \in (0,1)$. Closed-form:

$$\bar{k} = \left(\frac{s}{\delta}\right)^{\frac{1}{1-\alpha}}$$

Steady-state k/y ratio: $\bar{k}/\bar{y} = s/\delta$.

Dynamics and Convergence

Dynamics in the Solow model



Starting from k_0: use the curve to get k_1, place k_1 on horizontal axis, repeat.

Global, monotonic convergence to $\bar{k}$ regardless of k_0:

- If $k_t < \bar{k}$: $sf(k_t) > \delta k_t \Rightarrow k_{t+1} > k_t$ (capital rising)
- If $k_t > \bar{k}$: $sf(k_t) < \delta k_t \Rightarrow k_{t+1} < k_t$ (capital falling)

Intuition for Convergence

Rewrite the capital accumulation in growth-rate form:

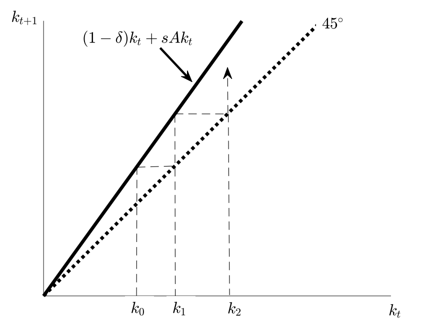
$$\frac{k_{t+1} - k_t}{k_t} = s\frac{f(k_t)}{k_t} - \delta$$

- $f(k)/k$ is **decreasing** in k (from $f''(k) < 0$ and $f(0) = 0$)
- When k is small: output per unit of capital is large $\Rightarrow$ high investment relative to capital $\Rightarrow$ capital grows fast
- When k is large: output per unit of capital is small $\Rightarrow$ depreciation dominates $\Rightarrow$ capital falls
- **Key**: decreasing returns to capital generate the convergence force

This de-mystifies the stability of k/y: deviations from $\bar{k}/\bar{y} = s/\delta$ are self-correcting.

Other Kinds of Dynamics: Endogenous Growth

Endogenous growth in the Solow model

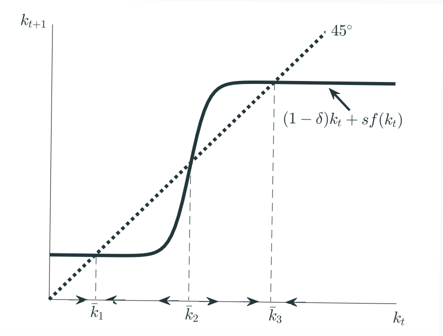


Special case: $y_t = Ak_t$ (linear in capital, $\alpha = 1$, no labor).

- Unbounded "endogenous" growth
- Two countries starting with different k_0 remain forever different (percentage gap constant)
- Empirically implausible: labor earns $\sim 2/3$ of income

Other Dynamics: Poverty Traps and Chaos

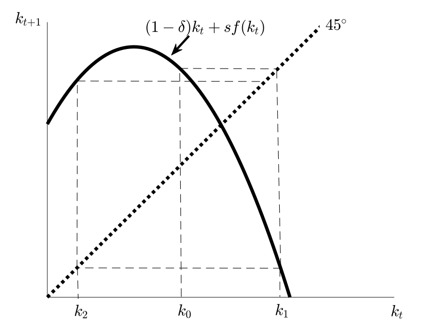
Poverty traps in the Solow model



- $\bar{k}_1$ and $\bar{k}_3$: stable; $\bar{k}_2$: unstable
- Economy starting with low k_0 converges to $\bar{k}_1$ (poverty trap)
- To escape: temporarily raise s enough to push past $\bar{k}_2$

Other Dynamics: Complex Dynamics

Complex dynamics in the Solow model



- Locally stable if slope at steady state is < 1 in absolute value
- These cases are more esoteric in macroeconomics but illustrate the range of possible dynamics

The Growing Economy

Adding Technology and Population Growth

$$Y_t = F(K_t, A_t L_t)$$

- A_t: labor-augmenting (Harrod-neutral) technology; $A_t L_t$: effective labor
- Growth rates: $A_{t+1}/A_t = 1 + \gamma$, $L_{t+1}/L_t = 1 + n$
- Uzawa (1961): labor-augmenting is the *only* form of technical change consistent with exact balanced growth

Define capital per effective worker: $\tilde{k}_t \equiv K_t/(A_t L_t)$.

Fundamental equation becomes:

$$(1+\gamma)(1+n)\tilde{k}_{t+1} = (1-\delta)\tilde{k}_t + sf(\tilde{k}_t)$$

Very similar to the basic model. Population growth n and technology growth γ act like additional "depreciation" on $\tilde{k}$.

Balanced Growth Path

Balanced growth path (BGP): $\tilde{k}_t$ is constant at $\bar{\tilde{k}}$:

$$(1+\gamma)(1+n)\bar{\tilde{k}} = (1-\delta)\bar{\tilde{k}} + sf(\bar{\tilde{k}})$$

Cobb-Douglas:

$$\bar{\tilde{k}} = \left(\frac{s}{(1+\gamma)(1+n)+\delta-1}\right)^{\frac{1}{1-\alpha}}$$

Convergence: Starting from any $\tilde{k}_0$, the sequence $\{\tilde{k}_t\}$ converges monotonically to $\bar{\tilde{k}}$.

On BGP: $Y_t/(A_t L_t) = f(\bar{\tilde{k}})$ is constant, so income per capita $y_t = Y_t/L_t = f(\bar{\tilde{k}})A_t$ grows at rate γ.

Long-Run Growth

Per capita income on the BGP:

$$y_t = f(\bar{\tilde{k}})A_t \qquad \Rightarrow \qquad \frac{y_{t+1} - y_t}{y_t} = \gamma$$

- **Long-run per capita growth rate** $= \gamma$ **(technology growth)**
- **No other parameters** affect the long-run growth rate
- Higher s raises the *level* of income, not the growth rate
- Higher s also raises the growth rate in the *short run* (during transition)

Short-run growth:

$$\frac{y_{t+1} - y_t}{y_t} = \frac{f(\tilde{k}_{t+1})}{f(\tilde{k}_t)}(1+\gamma) - 1$$

- If $\tilde{k}_t < \bar{\tilde{k}}$: short-run growth $> \gamma$ (poor countries grow faster)
- If $\tilde{k}_t > \bar{\tilde{k}}$: short-run growth $< \gamma$ (rich countries grow slower)

Stylized Facts and the Solow Model

Matching the Kaldor Facts

Constant K/Y: On BGP, $K_t/(A_t L_t) = \bar{\tilde{k}}$ and $Y_t/(A_t L_t) = f(\bar{\tilde{k}})$ are both constant, so K_t/Y_t is constant. ✓

Steady per capita growth: y_t grows at rate γ on BGP. ✓

Constant return to capital: Firms maximize profit under competition:

$$\max_{K_t, A_t L_t} F(K_t, A_t L_t) - r_t K_t - w_t A_t L_t$$

FOC: $r_t = F_1(K_t, A_t L_t) = f'(\tilde{k}_t)$. On BGP: $r_t = f'(\bar{\tilde{k}})$, which is constant. ✓

Factor Shares

Wage per efficiency unit: From the FOC for $A_t L_t$:

$$w_t = f(\tilde{k}_t) - \tilde{k}_t f'(\tilde{k}_t)$$

Constant on BGP (since $\tilde{k}_t = \bar{\tilde{k}}$).

Capital share: $r_t K_t / Y_t = \tilde{k}_t f'(\tilde{k}_t)/f(\tilde{k}_t)$. Constant on BGP. ✓

Labor share: $w_t A_t L_t / Y_t = 1-$ capital share (from CRS $\Rightarrow$ zero pure profits). Constant on BGP. ✓

Cobb-Douglas: $Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha}$

- Capital share $= \alpha$, labor share $= 1 - \alpha$, regardless of K_t or $A_t L_t$
- Factor shares constant even *off* the BGP

Convergence

Speed of Convergence: Basic Model

Linearize the fundamental equation around $\bar{k}$:

$$\Delta k_{t+1} = (1 - \delta + sf'(\bar{k}))\Delta k_t$$

where $\Delta k_t \equiv k_t - \bar{k}$.

Equivalently:

$$\frac{k_{t+1} - \bar{k}}{\bar{k}} = (1 - \lambda)\frac{k_t - \bar{k}}{\bar{k}}$$

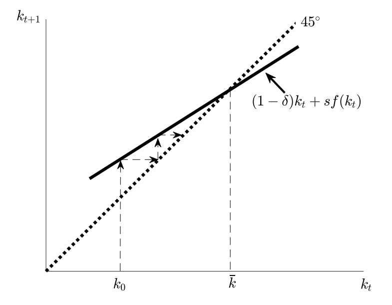
where $\lambda \equiv \delta - sf'(\bar{k})$ is the **convergence speed**.

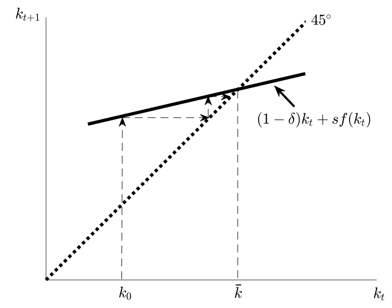
Cobb-Douglas: $\lambda = \delta(1 - \alpha)$.

- Faster convergence when α is small (stronger decreasing returns)
- Faster convergence when δ is large
- s does *not* affect λ in the Cobb-Douglas case (two opposing forces exactly cancel)

Speed of Convergence: Growing Economy

Slow and fast convergence





Higher λ = flatter slope at steady state = fewer steps to converge.

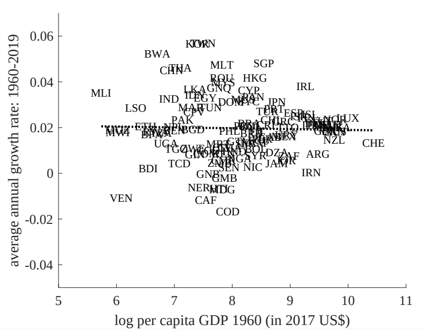
Linearizing the growing-economy fundamental equation:

$$\frac{\tilde{k}_{t+1} - \bar{\tilde{k}}}{\bar{\tilde{k}}} = (1-\lambda)\frac{\tilde{k}_t - \bar{\tilde{k}}}{\bar{\tilde{k}}}$$

Cobb-Douglas: $\lambda = (1-\alpha)\left(1 - \frac{1-\delta}{(1+\gamma)(1+n)}\right)$

Cross-Country Evidence: All Countries

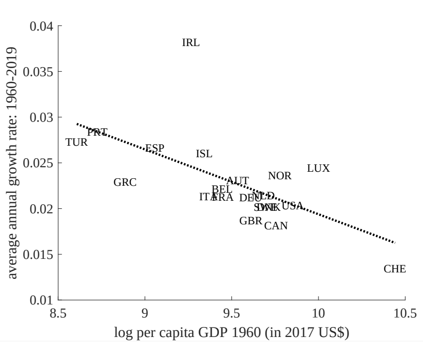
All countries, 1960–2019



- **No systematic tendency** for poor countries to grow faster
- Does not reject the Solow model: model predicts **conditional convergence** (countries converge if they share the same parameters), not *unconditional* convergence
- Saving rates, γ, and institutions differ widely across countries

Cross-Country Evidence: OECD Countries

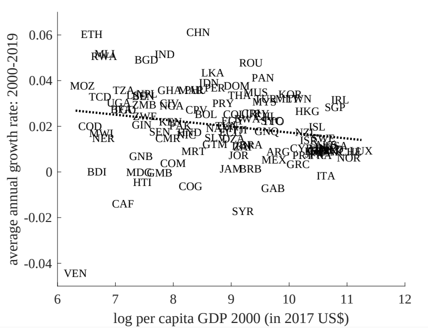
OECD countries, 1960–2019



- OECD: high-income countries with similar institutions
- **Clear tendency for convergence**: initially poorer OECD countries grew faster
- Barro and Sala-i-Martin (1995): similar patterns within US states, Japanese prefectures, European regions

Recent Evidence: Unconditional Convergence Emerging

All countries, 2000–2019



- Kremer, Willis, and You (2022): in recent years, data show a tendency for **unconditional** convergence
- Negative correlation between GDP and subsequent growth
- Explanation: underlying factors (policies, institutions, human capital) have become more similar across countries

Calibration

Calibrating the Solow Model

Goal: Assign functional forms and parameter values to obtain quantitative predictions.

Production function: Cobb-Douglas $F(K, AL) = K^{\alpha}(AL)^{1-\alpha}$

- Consistent with approximately constant factor shares
- Substitution elasticity ≈ 1 in aggregate production function studies

Parameters (annual frequency):

- $\alpha = 1/3$ (capital share from NIPA, Figure 2.12)
- $\gamma = 0.02$ (long-run per capita income growth)
- $n = 0.01$ (population growth)
- $\delta = 0.046$ (implied by $I/K \approx 0.076$ and the BGP condition $(1+\gamma)(1+n) = 1 - \delta + I/K$)

Quantitative Predictions: Convergence Speed

With $\alpha = 1/3$, $\gamma = 0.02$, $n = 0.01$, $\delta = 0.046$:

$$\lambda = (1-\alpha)\left(1 - \frac{1-\delta}{(1+\gamma)(1+n)}\right) = 0.049$$

- **Empirical estimate**: $\lambda \approx 0.015$ to 0.03 (Barro and Sala-i-Martin, 2004)
- **Model over-predicts** convergence speed
- To match $\lambda \approx 0.02$: need $\alpha \approx 0.73$
- Justification for larger α: part of labor income is return to **human capital** (accumulable, like physical capital); A_t itself may be accumulable (R&D investment)

Quantitative Experiments: Doubling the Saving Rate

With $s = 0.1$ and calibrated parameters:

- $\bar{\bar{k}} = 1.20$, $f(\bar{\bar{k}}) = 1.06$

Doubling to $s = 0.2$:

- $\bar{\bar{k}} = 1.90$, $f(\bar{\bar{k}}) = 1.24$

Result: Doubling the saving rate increases the normalized level of output by **17%** ($1.24/1.06 = 1.17$).

Can also compute quantitative predictions for how fast output rises to reach this new level (transition dynamics).

Business Cycles

The Solow Model as the Core of Business Cycle Theory

The Solow model's ability to account for long-run facts makes it the **core of macroeconomic modeling**. Virtually all modern business cycle theories build on it.

Business cycles = arrival of shocks + propagation mechanism.

Three types of shocks (extending the basic model):

1. Neutral technology shocks (RBC model)
2. Investment-specific technology shocks
3. Demand shocks (Keynesian)

In each case, k_{t+1} can be written as a function of k_t and the shock—a modified fundamental equation.

RBC Model: Technology Shocks

Production function with variable labor and TFP shock:

$$y_t = A_t F(k_t, \ell_t)$$

With endogenous saving and labor supply (to match business cycle facts):

$$k_{t+1} = (1-\delta)k_t + s(k_t, A_t) A_t F(k_t, \ell(k_t, A_t))$$

Business cycle facts require:

- ℓ_t comoves positively with cycle
- Investment i_t more volatile than y_t
- Consumption c_t less volatile than y_t

These cannot be matched with constant s and fixed ℓ—need endogenous responses to A_t.

Note: With Cobb-Douglas, Hicks-neutral ($A_t F(k, \ell)$) and Harrod-neutral ($F(k, A_t \ell)$) are equivalent:

Investment-Specific and Demand Shocks

Investment-specific shocks: Goods market becomes $y_t = c_t + i_t/\nu_t$, where ν_t varies. When ν_t is high, investment goods are cheaper:

$$k_{t+1} = s\nu_t F(k_t, \ell) + (1-\delta)k_t$$

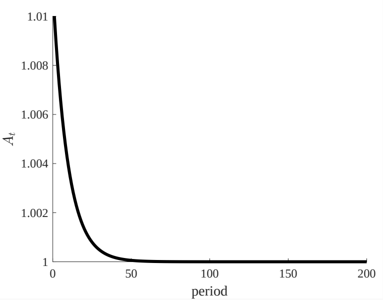
Demand shocks (Keynesian): Consumption c_t is exogenous, $i_t/c_t = s/(1-s)$. When demand is insufficient, unemployment u_t arises:

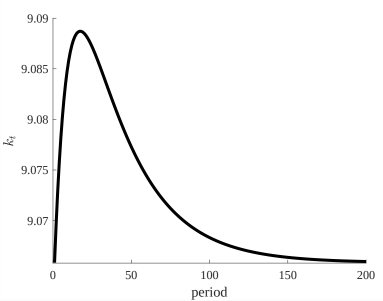
$$y_t = \frac{1}{1-s}c_t = F(k_t, \ell(1-u_t))$$
$$k_{t+1} = sF(k_t, \ell(1-u(c_t,k_t))) + (1-\delta)k_t$$

This is Keynesian in spirit but requires frictions to explain why output falls below full capacity. Several chapters address these frictions.

Impulse Response Functions

Impulse response—how A_t (left) affects k_t (right)





Experiment: $A_0 = (1+\varepsilon)\bar{A}$, $A_t = (1+\rho^t\varepsilon)\bar{A}$ for $t \geq 1$ in the baseline model

Three key features:

1. **Slow adjustment**: k_t response is far more persistent than A_t
2. **Small magnitude**: maximum deviation of k_t is only 0.25%
3. k_t **returns to steady state**: convergence force remains active

Log-Linearization

Express variables as percent deviations from steady state: $\hat{x}_t \equiv \log(x_t/\bar{x})$.

Log-linearizing the fundamental equation around $\bar{k}$:

$$\hat{k}_{t+1} = (1 - \delta(1-\alpha))\hat{k}_t + \delta\hat{A}_t = (1-\lambda)\hat{k}_t + \delta\hat{A}_t$$

With $\hat{A}_t = \rho^t \varepsilon$, the solution is:

$$\hat{k}_{t+1} = \rho^t \cdot \frac{1 - \left(\frac{1-\lambda}{\rho}\right)^{t+1}}{1 - \frac{1-\lambda}{\rho}} \cdot \delta\varepsilon$$

Log-linearized output response: $\hat{y}_t = \hat{A}_t + \alpha\hat{k}_t$.

Accuracy: Maximum error ≈ 0.00001 in levels—visually indistinguishable from the nonlinear solution.

Summary

Key Takeaways (I)

1. **Fundamental equation**: $k_{t+1} = (1-\delta)k_t + sf(k_t)$; unique steady state $\bar{k}$; global monotonic convergence
2. **Convergence force**: Decreasing returns to capital—when k is low, growth is fast; when k is high, growth is slow
3. **Growing economy**: Labor-augmenting technology is necessary for balanced growth (Uzawa); $\tilde{k}_t \to \bar{\tilde{k}}$; per capita growth rate $= \gamma$
4. **Stylized facts**: Matches constant K/Y, constant factor shares, constant r, rising wages, steady per capita growth

Key Takeaways (II)

5. **Convergence speed**: $\lambda = (1-\alpha)(1-(1-\delta)/((1+\gamma)(1+n)))$; model over-predicts unless α is raised (human capital)
6. **Cross-country data**: No unconditional convergence, but clear conditional convergence (OECD, US states); recent evidence of unconditional convergence post-2000
7. **Business cycles**: Solow model is the backbone; technology, investment-specific, and demand shocks; impulse responses; log-linearization
8. **Limitation**: Saving rate s and labor ℓ are exogenous; Chapter 4 will endogenize them via optimization