Chapter 4: Dynamic Optimization

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Introduction

Motivation

Solow model: saving rate s is exogenous. **This chapter**: endogenize saving via optimization.

Cass (1965) and Koopmans (1963): Added microeconomic foundations to the Solow model $\Rightarrow$ the **optimizing neoclassical growth model (NGM)**.

- Same long-run implications as Solow, but explicit utility $\Rightarrow$ welfare analysis
- Forward-looking decisions $\Rightarrow$ richer policy implications
- With stochastic shocks: foundation of real business cycle theory (Chapter 14)

Friedman (1957): Individuals are forward-looking—decisions depend on expected future income, not just current income.

Motivation

Two solution methods:

1. **Sequential methods**: maximize over sequences
2. **Recursive methods** (dynamic programming): solve functional equations

The Generic Problem

The Dynamic Optimization Problem

$$\max_{\{y_t, x_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t \hat{F}(y_t) \quad \text{s.t.} \quad x_{t+1} = h(x_t, y_t), \quad x_{t+1} \in \Gamma(x_t)$$

- y_t: **control variables** (consumption, hours, investment)
- x_t: **state variables** (capital, debt, housing stock)
- $\hat{F}$: per-period objective (utility or profit)
- β: discount factor; $1/\beta - 1$: discount rate
- h: law of motion linking states and controls
- $\Gamma(x_t)$: feasibility set
- x_0: given (exogenous initial condition)

Existence: Weierstrass theorem requires $\hat{F}$ continuous, Γ nonempty, closed, bounded.

Application 1: The Consumption-Saving Model

Agent lives T periods, maximizes $\sum_{t=0}^{T} \beta^t u(c_t)$.

Assumptions on u: strictly increasing, strictly concave, $\lim_{c \to 0} u'(c) = \infty$.

Budget constraint: $c_t = w_t + (1 + r_t)a_t - a_{t+1}$

Borrowing limit: $a_{t+1} \geq \underline{a}, \quad a_{T+1} \geq 0$

Mapping to generic form:

- Control: c_t; State: a_t
- $h(a_t, c_t) = w_t + (1 + r_t)a_t - c_t$
- $\Gamma(a_t) = [\underline{a},\ w_t + (1 + r_t)a_t]$

Application 2: The Neoclassical Growth Model

Same utility. Production: $y_t = f(k_t)$. Capital accumulation:

$$k_{t+1} = (1-\delta)k_t + i_t$$

Resource constraint (closed economy, $i_t = y_t - c_t$):

$$c_t + k_{t+1} \leq f(k_t) + (1-\delta)k_t$$

Planner's problem:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^T} \sum_{t=0}^{T} \beta^t u(c_t) \quad \text{s.t.} \quad c_t + k_{t+1} \leq f(k_t) + (1-\delta)k_t, \quad c_t, k_{t+1} \geq 0$$

Mapping: Control: c_t; State: k_t; $\Gamma(k_t) = [0, f(k_t) + (1-\delta)k_t]$.

Chapter 6: Under perfect competition, the competitive equilibrium solves this planning problem (welfare theorems).

Sequential Methods: Finite Horizon

Two-Period Consumption-Saving Model

$T = 2$, $a_0 = 0$, $\underline{a} = 0$, constant r. Budget constraints combine into the **lifetime budget constraint**:

$$c_0 + \frac{c_1}{1+r} = w_0 + \frac{w_1}{1+r}$$

Lagrangian: $\mathcal{L} = u(c_0) + \beta u(c_1) + \mu[\text{LBC}] + \lambda[w_0 - c_0]$

Interior solution ($\lambda = 0$) yields the **Euler equation**:

$$u'(c_0) = \beta(1+r)u'(c_1)$$

- LHS: marginal cost of saving one unit (less consumption today)
- RHS: discounted marginal benefit ($(1+r)$ units tomorrow, valued at $\beta u'(c_1)$ each)
- When $\beta(1+r) = 1$: full smoothing, $c_0 = c_1 = (w_0 + \beta w_1)/(1+\beta)$

Generic T-Period Model: Result 1

Substitute the constraint into the objective:
$F(x_t, x_{t+1}) \equiv \hat{F}(\hat{h}(x_t, x_{t+1}))$.

Result 1: If F is jointly concave in (x_t, x_{t+1}), increasing in x_t, decreasing in x_{t+1}, and the interior conditions hold, then the optimum satisfies:

Euler equation:

$$F_2(x_t^*, x_{t+1}^*) + \beta F_1(x_{t+1}^*, x_{t+2}^*) = 0 \qquad \forall t < T$$

Terminal condition: $x_{T+1}^* = \underline{x}$

- $T + 2$ equations (Euler + initial + terminal), $T + 2$ unknowns
- Second-order difference equation in the state variable
- "Variational" condition: given x_t and x_{t+2}, vary x_{t+1} to achieve the best outcome

Finite-Horizon NGM: Euler Equation

Substituting the resource constraint into utility:

Lagrangian: $\mathcal{L} = \sum_{t=0}^{T} \beta^t \{u(f(k_t) + (1-\delta)k_t - k_{t+1}) + \mu_t k_{t+1}\}$

FOC ($t = 0, \ldots, T-1$):

$$u'(c_t) = \beta u'(c_{t+1})[f'(k_{t+1}) + 1 - \delta] + \mu_t$$

Terminal period: $u'(c_T) = \mu_T > 0 \Rightarrow k_{T+1} = 0$.

Interior Euler equation (Inada $\Rightarrow k_{t+1} > 0$ for $t < T$, so $\mu_t = 0$):

$$u'(c_t) = \beta u'(c_{t+1})[f'(k_{t+1}) + 1 - \delta]$$

Three components: marginal cost of investment = utility per unit of return $\times$ return on investment.

Solving by Backward Induction (Example 4.1)

Log utility, Cobb-Douglas $f(k) = Ak^{\alpha}$, full depreciation $\delta = 1$.

Start from $k_{T+1} = 0$, use Euler equation backwards:

$$k_T = \frac{\alpha\beta}{1+\alpha\beta} A k_{T-1}^{\alpha}, \quad k_{T-1} = \frac{\alpha\beta(1+\alpha\beta)}{1+\alpha\beta+(\alpha\beta)^2} A k_{T-2}^{\alpha}, \quad \ldots$$

General solution:

$$k_{t+1} = \alpha\beta \cdot \frac{1-(\alpha\beta)^{T-t}}{1-(\alpha\beta)^{T-t+1}} \cdot A k_t^{\alpha}$$

Saving rate: $s_t = \alpha\beta \cdot \frac{1-(\alpha\beta)^{T-t}}{1-(\alpha\beta)^{T-t+1}}$ — depends on time (distance to T), but not on k_t.

CRRA Utility and the EIS

Constant Relative Risk Aversion (CRRA) utility:

$$u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma},$$

$\sigma > 0$, $\sigma \neq 1$. Special cases: $\sigma = 0$ (linear), $\sigma \to 1$ (log), $\sigma \to \infty$ (Leontief).

Elasticity of Intertemporal Substitution (EIS): How does c_{t+s}/c_t respond to a change in the return $R_{t,t+s}$?

With CRRA and linear production $f_t(k) = R_t k$:

$$\frac{c_{t+s}}{c_t} = (\beta^s)^{1/\sigma}(R_{t,t+s})^{1/\sigma} \qquad \Rightarrow \qquad \text{EIS} = \frac{\partial \log(c_{t+s}/c_t)}{\partial \log R_{t,t+s}} = \frac{1}{\sigma}$$

- $\sigma < 1$: substitution effect dominates; investment rises with returns
- $\sigma > 1$: income effect dominates; investment falls with returns
- $\sigma = 1$ (log): income and substitution effects exactly cancel

Sequential Methods: Infinite Horizon

Moving to Infinite Horizon

Advantage: Problem becomes **stationary—**the maximization at t is the same as at $t + 1$ for the same capital level.

Dynastic interpretation: β^t = weight on descendants t generations ahead. $\beta < 1$: care more about self than offspring.

Mathematical complications:

1. **Unbounded utility**: Need $\beta < 1$, and with growth at rate γ, need $\beta(1 + \gamma)^{1-\sigma} < 1$
2. **Constraint sets "too large"**: Need borrowing limits to prevent infinite debt accumulation

The No-Ponzi-Game Condition

Without a terminal period, need to rule out schemes where agent borrows and never repays.

Consider an example with the period budget constraint $c_t + a_{t+1} = (1+r)a_t$.

No-Ponzi-game (nPg) condition:

$$\lim_{t\to\infty} \frac{a_{t+1}}{(1+r)^t} \geq 0$$

Agent cannot have positive debt (in present value) at infinity.

Lifetime budget constraint

$$\sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t} \leq a_0(1+r)$$

The nPg condition and the lifetime budget constraint (and the natural borrowing limit, in this case $a_t \geq 0$ for all t) are equivalent.

The Transversality Condition

The transversality condition (TVC):

$$\lim_{t\to\infty} \frac{a_{t+1}}{(1+r)^t} \leq 0$$

Interpretation: It is never optimal to hold positive assets (in present value) at infinity—better to consume them.

nPg vs. TVC:

- **nPg**: Externally imposed. Prevents over-*borrowing*.
- **TVC**: Self-imposed optimality condition.

In the NGM, the TVC takes the form:

$$\lim_{t\to\infty} \beta^t u'(c_t)[f'(k_t) + 1 - \delta]k_t = 0$$

If the economy converges to a steady state with $\beta < 1$: TVC satisfied automatically.

Sufficiency: Proposition 4.4

Proposition: If $F(x_t, x_{t+1})$ is jointly concave, Γ is convex, and x_t^* satisfies:

(i) Euler equation: $F_2(x_t^*, x_{t+1}^*) + \beta F_1(x_{t+1}^*, x_{t+2}^*) = 0$ for all t

(ii) TVC: $\lim_{t\to\infty} \beta^t F_1(x_t^*, x_{t+1}^*) x_t^* = 0$

then $\{x_t^*\}$ maximizes the objective.

Proof strategy (Appendix 4.A.3): Use the Euler equation repeatedly plus the definition of concavity $\left(f(x) \leq f(x^*) + f'(x^*)(x - x^*)\right)$ to show that any alternative feasible sequence gives a lower objective value—provided the TVC holds.

Solving the Infinite-Horizon NGM (Example 4.5)

Log utility, $f(k) = Ak^{\alpha}$, $\delta = 1$. Take the limit of the finite-horizon solution as $T \to \infty$:

$$k_{t+1} = \lim_{T\to\infty} \alpha\beta \cdot \frac{1-(\alpha\beta)^{T-t}}{1-(\alpha\beta)^{T-t+1}} \cdot Ak_t^{\alpha} = \alpha\beta Ak_t^{\alpha}$$

- Saving rate $s = \alpha\beta$ (constant—no longer depends on time!)
- Stationarity: infinite horizon $\Rightarrow$ same problem every period
- TVC: $\lim_{t\to\infty} \beta^t \alpha/(1-\alpha\beta) = 0$ since $\beta < 1$ ✓

Steady state: $\bar{k} = (\beta\alpha A)^{1/(1-\alpha)}$ — identical to Solow with $s = \alpha\beta$ and $\delta = 1$.

Guess and verify (Example 4.6): Guess $k_{t+1} = sAk_t^{\alpha}$, substitute into Euler equation $\Rightarrow s = \alpha\beta$.

Steady State and Balanced Growth

Steady state (no growth): From the Euler equation with $c_t = c_{t+1}$:

$$1 = \beta(f'(\bar{k}) + 1 - \delta)$$

Unique solution under Inada conditions. This is the **modified golden rule**: $f'(\bar{k}) = 1/\beta - 1 + \delta$.

Balanced growth: With $Y_t = F(K_t, A_t L_t)$, CRRA utility, define $\tilde{k}_t = K_t/(A_t L_t)$.

Adjusted discount factor: $\tilde{\beta} \equiv \beta(1+n)(1+\gamma)^{1-\sigma}$

Euler equation on BGP:

$$(1+n)(1+\gamma) = \tilde{\beta}[f'(\bar{\tilde{k}}) + 1 - \delta]$$

CRRA is the only utility consistent with balanced growth.

Recursive Methods: Dynamic Programming

The Bellman Equation

Key idea: Break the infinite-dimensional problem into a sequence of identical small problems.

Define the **value function**:

$$V(x_0) = \max_{\{x_{t+1} \in \Gamma(x_t)\}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$$

Stationarity allows us to write:

$$V(x) = \max_{x' \in \Gamma(x)} \{F(x, x') + \beta V(x')\}$$

This is the **Bellman equation**—a functional equation (unknown is V, a function).

Policy function: $g(x) = \arg\max_{x' \in \Gamma(x)} \{F(x, x') + \beta V(x')\}$

$\Rightarrow x' = g(x)$: stationary decision rule.

The NGM in Recursive Form

The standard NGM:

$$V(k) = \max_{k' \in [0, f(k) + (1-\delta)k]} \{u(f(k) + (1-\delta)k - k') + \beta V(k')\}$$

State: k (current capital). **Control**: k' (or equivalently c).

Other examples:

- **Periodic NGM** (alternating TFP A_h, A_l): Two value functions $V_h(k), V_l(k)$.
- **Delayed depreciation NGM** (capital depreciates after two periods): Two state variables (i_{-1}, i_{-2}).

Key lesson: Choosing state variables appropriately is essential for writing the problem recursively.

Properties of the Value Function

Under standard assumptions (F continuous, concave; Γ nonempty, compact, convex; $\beta < 1$):

1. **Contraction mapping**: Starting from *any* V_0, iterating $V_{n+1}(x) = \max_{x'}\{F(x, x') + \beta V_n(x')\}$ converges to V. Rate: $\|V_{n+1} - V\| \leq \beta\|V_n - V\|$.
2. V **is strictly increasing** (if F increasing in first argument and Γ monotone)
3. V **is strictly concave** (if F jointly strictly concave and Γ convex) $\Rightarrow$ policy g is unique
4. V **is differentiable** at interior solutions (if F is C^1)
5. g **is strictly increasing** (under all above conditions)

Property 1 is crucial for numerical computation (value function iteration, Chapter 10).

Solving the Bellman Equation (Example 4.7)

Log utility, $f(k) = Ak^\alpha$, $\delta = 1$. Start from $V_0(k) = 0$:

$V_1(k) = \max_{k'} \log(Ak^\alpha - k') = \log A + \alpha \log k$ (optimal: $k' = 0$)

$V_2(k) = \max_{k'} \{\log(Ak^\alpha - k') + \beta[\log A + \alpha \log k']\}$

$\Rightarrow k' = \alpha\beta A k^\alpha / (1 + \alpha\beta)$

Pattern: $V_n(k) = a_n + b_n \log k$ for all n.

Guess and verify: Guess $V(k) = a + b \log k$. Solution:

$$b = \frac{\alpha}{1 - \alpha\beta}, \qquad a = \frac{1}{1-\beta}\frac{1}{1-\alpha\beta}\left[\log A + \log(1-\alpha\beta)^{1-\alpha\beta}(\alpha\beta)^{\alpha\beta}\right]$$

Policy: $k' = \alpha\beta A k^\alpha$ — same answer as sequential methods.

The Functional Euler Equation and the Envelope Theorem

FOC of Bellman equation: $-F_2(x, g(x)) = \beta V'(g(x))$

Envelope Theorem: $V'(x) = F_1(x, g(x))$ (indirect effect through optimal x' vanishes at optimum)

Updating: $V'(g(x)) = F_1(g(x), g(g(x)))$

Functional Euler equation:

$$F_2(x, g(x)) + \beta F_1(g(x), g(g(x))) = 0 \qquad \forall x$$

- Same as the sequential Euler equation, but written as a **functional equation** in g
- Unknown is the function g (not a sequence)
- Necessary condition only (may have multiple solutions); the Bellman equation selects the correct one
- Useful for numerical computation and for proving convergence

Global Convergence in the NGM

Proof of Global Convergence

Goal: Show that $k_t \to \bar{k}$ from any $k_0 > 0$.

Use strict concavity of $V \Rightarrow [V'(k) - V'(g(k))][k - g(k)] \leq 0$ for all k.

Envelope Theorem: $V'(k) = u'(c)[f'(k) + 1 - \delta]$

FOC: $V'(g(k)) = u'(c)/\beta$

Substituting:

$$u'(c)\left[f'(k) + 1 - \delta - \frac{1}{\beta}\right][k - g(k)] \leq 0$$

- $k = \bar{k}$: $f'(\bar{k}) + 1 - \delta = 1/\beta$ and $g(\bar{k}) = \bar{k}$ ✓
- $k > \bar{k}$: $f'(k) + 1 - \delta < 1/\beta \Rightarrow g(k) < k$ (capital falls)
- $k < \bar{k}$: $f'(k) + 1 - \delta > 1/\beta \Rightarrow g(k) > k$ (capital rises)

Combined with g strictly increasing: **global, monotonic convergence** to $\bar{k}$.

Local Approximation of the Policy Function

Approximate $g(k) \approx a_0 + a_1 k$ near $\bar{k}$.

From $g(\bar{k}) = \bar{k}$: $a_0 = \bar{k}(1 - a_1)$.

Differentiate the functional Euler equation w.r.t. k, evaluate at $\bar{k}$:

$$u''(\bar{c})[1 - \beta a_1] = u''(\bar{c})[a_1 - \beta a_1^2] + \beta^2 u'(\bar{c}) f''(\bar{k}) a_1$$

Define $\Theta \equiv \beta^2 u'(\bar{c}) f''(\bar{k}) / u''(\bar{c}) > 0$. Solving the quadratic:

$$a_1 = \frac{1 + \beta + \Theta - \sqrt{(1 + \beta + \Theta)^2 - 4\beta}}{2\beta}$$

- $a_1 \in (0, 1]$: the correct root (monotone convergence)
- a_1 decreasing in Θ (faster convergence when Θ is large)
- σ large (unwilling to change c) $\Rightarrow \Theta$ small $\Rightarrow a_1$ close to 1 $\Rightarrow$ slow convergence
- $\sigma = 0$ (linear utility) $\Rightarrow \Theta \to \infty \Rightarrow a_1 \to 0 \Rightarrow$ immediate convergence

Calibration: The Role of σ

With CRRA $u(c) = (c^{1-\sigma} - 1)/(1-\sigma)$ and Cobb-Douglas $f(k) = k^\alpha$:

$$\Theta = \frac{\beta^2(1-\alpha)}{\sigma}\left(\frac{1}{\beta} - 1 + \delta\right)\left(\frac{1}{\alpha}\left(\frac{1}{\beta} - 1 + \delta\right) - \delta\right)$$

Θ is positive, decreasing in σ, decreasing in α, increasing in δ.

Empirical estimates of $1/\sigma$ (EIS):

- Hall (1988), aggregate data: $1/\sigma \approx 0.1$
- Attanasio and Weber (1993, 1995), micro data: $1/\sigma \in [0.3, 0.8]$
- Most applied macro: $\sigma = 1$ (log) or $\sigma = 2$

Result: Convergence in the NGM is qualitatively like Solow but quantitatively different—the endogenous saving rate depends on k through σ.

Summary

Key Takeaways (I)

1. **Generic problem**: Controls y_t, states x_t, discount factor β, feasibility Γ
2. **Euler equation**: $F_2(x_t, x_{t+1}) + \beta F_1(x_{t+1}, x_{t+2}) = 0$ — marginal cost of saving = marginal benefit
3. **Terminal condition** (finite horizon): $x_{T+1} = \underline{x}$ (consume everything)
4. **nPg condition** (infinite horizon): $\lim a_{t+1}/(1+r)^t \geq 0$ (no exploding debt)
5. **TVC**: $\lim \beta^t F_1 x_t = 0$ (no over-accumulation); self-imposed optimality condition

Key Takeaways (II)

6. **CRRA utility**: EIS $= 1/\sigma$; the only utility consistent with balanced growth
7. **Bellman equation**: $V(x) = \max_{x'}\{F(x, x') + \beta V(x')\}$; contraction mapping $\Rightarrow$ unique V, can be found by iteration
8. **Properties of V**: increasing, strictly concave, differentiable at interior; policy g unique and increasing
9. **Functional Euler equation**: $F_2(x, g(x)) + \beta F_1(g(x), g(g(x))) = 0$; necessary condition from Envelope Theorem
10. **Global convergence**: Proved using strict concavity of V + Envelope Theorem; speed depends on σ (curvature of utility) and α (curvature of production)
11. **Local approximation**: $g(k) \approx a_0 + a_1 k$; a_1 solved from a quadratic; $a_1 \in (0, 1)$ for the stable root