Chapter 5 Dynamic Competitive Equilibrium

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Introduction

What Is a Competitive Equilibrium?

Two conditions must hold simultaneously:

1. Agents maximize their objectives subject to constraints, taking prices as given
2. Markets clear: demand equals supply for all traded goods

Three formulations:

1. **Arrow-Debreu**: all trades at date 0; prices $\{p_t\}_{t=0}^{\infty}$
2. **Sequential**: trades period by period; bond price q_t each period
3. **Recursive**: prices and policies as functions of state variables

For many economies, all three yield the **same allocations**. The first two express equilibria as sequences; the third as functions.

Arrow-Debreu Equilibrium

Endowment Economy: Primitives

- Continuum of consumers $i \in [0,1]$; each produces $y_{i,t} \in \mathbb{R}_+$ of a single non-storable good
- Common preferences:

  $$\sum_{t=0}^{\infty} \beta^t u(c_{i,t}),$$

  with u strictly increasing and strictly concave
- Price p_t: price of time-t good in terms of time-0 good; normalize $p_0 = 1$
- Budget constraint (equality since u strictly increasing):

  $$\sum_{t=0}^{\infty} p_t c_{i,t} = \sum_{t=0}^{\infty} p_t y_{i,t}$$

Definition 1

An Arrow-Debreu competitive equilibrium is a set of sequences $\{c^*_{i,t}\}_{t=0}^{\infty}$, for each $i \in [0,1]$, and $\{p_t\}_{t=0}^{\infty}$ such that

1. for each i, $\{c^*_{i,t}\}_{t=0}^{\infty}$ solves

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \text{s.t.} \quad \sum_{t=0}^{\infty} p_t c_t = \sum_{t=0}^{\infty} p_t y_{i,t}$$

2. for each t,

$$\int_0^1 c^*_{i,t}\, di = \int_0^1 y_{i,t}\, di$$

Point 1: optimization. Point 2: goods market clearing.

Characterization: Prices

FOC from the Lagrangian on the lifetime budget:

$$\beta^t u'(c_{i,t}) = \lambda_i p_t$$

where λ_i is the multiplier on the lifetime budget constraint. It does not depend on t.

Eliminating λ_i between $t = 0$ and a generic t:

$$p_t = \beta^t \frac{u'(c_{i,t})}{u'(c_{i,0})}$$

Since p_t is common to all agents: the ratio of marginal utilities at different dates is **equalized across all consumers**.

Log Utility: Prices and Consumption Growth

With $u(\cdot) = \log(\cdot)$: $p_t = \beta^t c_{i,0}/c_{i,t}$. Multiply by $c_{i,t}$, integrate over i, use market clearing ($\int_0^1 c_{i,t}\, di = Y_t$):

$$p_t = \beta^t \frac{Y_0}{Y_t}$$

All consumers share the same consumption growth:

$$\frac{c_{i,t+1}}{c_{i,t}} = \frac{Y_{t+1}}{Y_t} = \beta \frac{p_t}{p_{t+1}} \quad \text{for all } i$$

If $Y_t = \bar{Y}$ constant: $c_{i,t}$ is constant for all i and t (**perfect consumption smoothing**).

Consumption Levels and Shares

From the Euler equations and the budget constraint:

$$c_{i,0} = (1-\beta) Y_0 \sum_{t=0}^{\infty} \beta^t \frac{y_{i,t}}{Y_t}$$

Individual i's share of aggregate consumption is constant over time:

$$c_{i,t} = \theta_i Y_t, \qquad \theta_i = (1-\beta) \sum_{t=0}^{\infty} \beta^t \frac{y_{i,t}}{Y_t}$$

Endowments in periods with high Y_t receive lower weight (marginal utility is lower). Two consumers with the same average endowments can have different wealth.

General CRRA: $p_t = \beta^t (Y_0/Y_t)^\sigma$. Higher $\sigma \Rightarrow$ resources more valuable in low-endowment periods.

Definition 2 (Representative Agent)

When all agents have equal endowments and u is strictly concave, all consumption choices are identical:

An AD CE is $\{c_t^*\}_{t=0}^{\infty}$ and $\{p_t\}_{t=0}^{\infty}$ such that

1. $\{c_t^*\}_{t=0}^{\infty}$ solves

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \text{s.t.} \quad \sum_{t=0}^{\infty} p_t c_t = \sum_{t=0}^{\infty} p_t y_t$$

2. $c_t^* = y_t$

Production Economy with Labor: Primitives

- Household i: efficiency units e_i, hours $\ell_{i,t}$, utility

$$\sum_{t=0}^{\infty} \beta^t [u(c_{i,t}) - v(\ell_{i,t})],$$

  with v increasing and convex
- Firm: $Y_t = A_t L_t$ (CRS), hires labor at wage w_t per efficiency unit
- Firm profit max:

$$\max_{\{L_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \{p_t A_t L_t - p_t w_t L_t\}$$

- Household budget:

$$\sum_{t=0}^{\infty} p_t c_{i,t} = \sum_{t=0}^{\infty} p_t w_t e_i \ell_{i,t}$$

Definition 3

An AD CE is $\{c^*_{i,t}\}_{t=0}^\infty$, $\{\ell^*_{i,t}\}_{t=0}^\infty$ for each $i \in [0,1]$, $\{L^*_t\}_{t=0}^\infty$, $\{p_t\}_{t=0}^\infty$, $\{w_t\}_{t=0}^\infty$ such that

1. for each i, $\{c^*_{i,t}\}_{t=0}^\infty$ and $\{\ell^*_{i,t}\}_{t=0}^\infty$ solve

$$\max_{\{c_t,\ell_t\}_{t=0}^\infty} \sum_{t=0}^\infty \beta^t[u(c_t) - v(\ell_t)] \quad \text{s.t.} \quad \sum_{t=0}^\infty p_t c_t = \sum_{t=0}^\infty p_t w_t e_i \ell_t$$

2. $\{L^*_t\}_{t=0}^\infty$ solves

$$\max_{\{L_t\}_{t=0}^\infty} \sum_{t=0}^\infty \{p_t A_t L_t - p_t w_t L_t\}$$

3. for each t,

$$\int_0^1 e_i \ell^*_{i,t} \, di = L^*_t$$

and

$$\int_0^1 c^*_{i,t} \, di = A_t L^*_t$$

Characterization of Definition 3

Household FOCs:

$$\beta^t u'(c_{i,t}) = \lambda_i p_t, \qquad v'(\ell_{i,t}) = e_i w_t u'(c_{i,t})$$

Firm FOC: For an interior solution with CRS, $w_t = A_t$. If $A_t > w_t$: no finite optimum. If $A_t < w_t$: firm shuts down.

With log utility and $w_t = A_t$: the intratemporal FOC becomes $v'(\ell_{i,t})c_{i,t} = e_i A_t$. With $c_{i,t} = \theta_i Y_t$ and conjecturing $\theta_i = e_i$: all agents supply the same labor $\ell_{i,t} = L_t$. Income and substitution effects exactly cancel.

With heterogeneous initial assets: labor supply depends on wealth nonlinearly $\Rightarrow$ aggregate output depends on the distribution of assets.

Neoclassical Growth Economy: Primitives

- Production:

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$$

- Capital:

$$k_{i,t+1} = (1-\delta)k_{i,t} + \iota_{i,t}$$

- Firm:

$$\max_{\{K_t, L_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \{p_t A_t K_t^{\alpha} L_t^{1-\alpha} - p_t r_t K_t - p_t w_t L_t\}$$

- Household budget:

$$\sum_{t=0}^{\infty} p_t (c_{i,t} + \iota_{i,t}) = \sum_{t=0}^{\infty} p_t (r_t k_{i,t} + w_t e_i \ell_{i,t})$$

Definition 4

An AD CE is $\{c_{i,t}^*\}_{t=0}^\infty$, $\{\iota_{i,t}^*\}_{t=0}^\infty$, $\{\ell_{i,t}^*\}_{t=0}^\infty$ for each i, $\{L_t^*\}_{t=0}^\infty$, $\{K_{t+1}^*\}_{t=0}^\infty$, $\{p_t\}_{t=0}^\infty$, $\{p_t w_t\}_{t=0}^\infty$, $\{p_t r_t\}_{t=0}^\infty$ such that

1. for each i, $\{c_{i,t}^*\}_{t=0}^\infty$, $\{\iota_{i,t}^*\}_{t=0}^\infty$, $\{\ell_{i,t}^*\}_{t=0}^\infty$ solve

$$\max_{\{c_t,\iota_t,\ell_t\}_{t=0}^\infty} \sum_{t=0}^\infty \beta^t [u(c_t) - v(\ell_t)]$$

$$\text{s.t.} \quad \sum_{t=0}^\infty p_t(c_t + \iota_t) = \sum_{t=0}^\infty p_t(r_t k_t + w_t e_i \ell_t)$$

   where $k_{t+1} = (1-\delta)k_t + \iota_t$, $k_0 = k_{i,0}$

2. $\{L_t^*\}_{t=0}^\infty$, $\{K_t^*\}_{t=0}^\infty$ (with $K_0^* = \int_0^1 k_{i,0}\,di$) solve

$$\max_{\{K_t,L_t\}_{t=0}^\infty} \sum_{t=0}^\infty \left\{p_t A_t K_t^\alpha L_t^{1-\alpha} - p_t r_t K_t - p_t w_t L_t\right\}$$

3. $\int_0^1 e_i \ell_{i,t}^*\,di = L_t^*$, $\int_0^1 k_{i,t}^*\,di = K_t^*$,
   $\int_0^1 (c_{i,t}^* + \iota_{i,t}^*)\,di = A_t (K_t^*)^\alpha (L_t^*)^{1-\alpha}$

Optimality Conditions for Definition 4

Consumption and labor (same structure as before):

$$\beta^t u'(c_{i,t}) = \lambda_i p_t, \qquad v'(\ell_{i,t}) = e_i w_t u'(c_{i,t})$$

Investment (new): the household's FOC for $\iota_{i,t}$ yields

$$p_t = p_{t+1} r_{t+1} + (1-\delta) p_{t+1} = p_{t+1}[r_{t+1} + 1 - \delta]$$

Cost of one unit of capital today = rental revenue + undepreciated value tomorrow.

Firm FOCs: $r_t = \alpha A_t K_t^{\alpha-1} L_t^{1-\alpha}$, $w_t = (1-\alpha) A_t K_t^{\alpha} L_t^{-\alpha}$. CRS $\Rightarrow$ zero profits.

Equivalence with the Planner

With identical agents ($e_i = 1$, $k_{i,0} = K_0$) ad log utility, the aggregate equilibrium conditions become:

$$\frac{C_{t+1}}{C_t} = \beta[1 - \delta + \alpha A_{t+1} K_{t+1}^{\alpha-1} L_{t+1}^{1-\alpha}]$$

$$v'(L_t) = \frac{(1-\alpha) A_t K_t^{\alpha} L_t^{-\alpha}}{C_t}$$

$$C_t + K_{t+1} = A_t K_t^{\alpha} L_t^{1-\alpha} + (1-\delta) K_t$$

These are identical to the FOCs of the social planner from Chapter 4 ⇒ **First Welfare Theorem**.

Sequential Equilibrium

Sequential Trading: Setup

- Trades occur period by period; bond price q_t (pay q_t today, receive 1 tomorrow)
- Gross interest rate:

$$1 + r_{t+1} = 1/q_t$$

- Period budget:

$$c_{i,t} + q_t a_{i,t+1} = y_{i,t} + a_{i,t}$$

- Need a no-Ponzi-game condition to rule out infinite borrowing
- Initial assets:

$$\int_0^1 a_{i,0}\, di = 0$$

Definition 5

A sequential CE is $\{c^*_{i,t}\}_{t=0}^{\infty}$, $\{a^*_{i,t+1}\}_{t=0}^{\infty}$ for each $i \in [0,1]$, and $\{q_t\}_{t=0}^{\infty}$ such that

1. for each i, $\{c^*_{i,t}\}_{t=0}^{\infty}$ and $\{a^*_{i,t+1}\}_{t=0}^{\infty}$ solve

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

   s.t. $c_t + q_t a_{t+1} = a_t + y_{i,t}$ for all t, with $a_0 = a_{i,0}$,

$$\lim_{t \to \infty} \left(\prod_{s=0}^{t} q_s \right) a_{t+1} \geq 0 \quad \text{(nPg)}$$

2. for all t,

$$\int_0^1 c^*_{i,t}\, di = \int_0^1 y_{i,t}\, di \text{ and } \int_0^1 a^*_{i,t+1}\, di = 0$$

The second clearing condition says aggregate assets net out to zero. By Walras' Law, only one clearing condition is needed.

Sequential FOCs

FOCs from the Lagrangian with period-by-period budget constraints:

$$\beta^t u'(c_{i,t}) = \lambda_{i,t}, \qquad q_t \lambda_{i,t} = \lambda_{i,t+1}$$

Unlike the AD case, $\lambda_{i,t}$ now depends on t (one multiplier per period budget).

Eliminating multipliers:

$$q_t = \beta \frac{u'(c_{i,t+1})}{u'(c_{i,t})}$$

Equivalence: Sequential ⇔ Arrow-Debreu

Chaining from 0 to $t-1$:

$$\prod_{s=0}^{t-1} q_s = \beta^t \frac{u'(c_{i,t})}{u'(c_{i,0})} = p_t$$

The product of sequential bond prices equals the Arrow-Debreu price. Also: $p_{t-1}/p_t = 1/q_{t-1} = 1 + r_t$.

Consolidating period budgets forward and using TVC (so the asset term vanishes):

$$\sum_{t=0}^{\infty} p_t c_{i,t} = a_{i,0} + \sum_{t=0}^{\infty} p_t y_{i,t}$$

Same lifetime budget ⇒ same allocations.

Definition 6 (Sequential CE, NGM)

A sequential CE is $\{c^*_{i,t}\}_{t=0}^\infty$, $\{k^*_{i,t+1}\}_{t=0}^\infty$ for each $i \in [0,1]$, $\{r_t\}_{t=0}^\infty$, $\{w_t\}_{t=0}^\infty$ such that

1. for each i, $\{c^*_{i,t}\}_{t=0}^\infty$ and $\{k^*_{i,t+1}\}_{t=0}^\infty$ solve
$$\max_{\{c_t,k_{t+1}\}_{t=0}^\infty} \sum_{t=0}^\infty \beta^t u(c_t)$$
s.t. $c_t + k_{t+1} = (1-\delta+r_t)k_t + w_t e_i$ for all t, with $k_0 = k_{i,0}$,
$$\lim_{t\to\infty} \frac{k_{t+1}}{\prod_{s=0}^t (1-\delta+r_{s+1})} \geq 0 \quad \text{(nPg)}$$
2. for each t, $(K_t, 1)$ solves $\max_{K,L} A_t K^\alpha L^{1-\alpha} - r_t K - w_t L$, with $K_t = \int_0^1 k^*_{i,t}\, di$
3. for each t, $C_t + K_{t+1} = A_t K_t^\alpha + (1-\delta)K_t$, where $C_t = \int_0^1 c^*_{i,t}\, di$

By Walras' Law, condition 3 is redundant given condition 2

Recursive Equilibrium

Recursive Methods: Overview

Express equilibrium as **functions** of state variables, not sequences indexed by time.

A state variable must be both **relevant** (determines prices) and **predetermined**.

Two steps:

1. **Steady-state** recursive equilibria: aggregate variables constant; functions specify that agents *choose* to remain at constant levels
2. **Dynamic** recursive equilibria: aggregates change with the state K; agents use K to forecast future prices via $r(K)$, $w(K)$, $K' = G(K)$

Definition 7 (Steady-State CE, Endowment Economy)

With constant endowments y_i for each i. A recursive steady-state CE is a q and a set of functions $V_i^*(a)$ and $g_i^*(a)$, and asset values a_i^*, for each $i \in [0,1]$, such that

1. for each i, $V_i^*(a)$ solves

$$V_i(a) = \max_{a'} \left\{ u(a + y_i - qa') + \beta V_i(a') \right\} \quad \forall a$$

   with $g_i^*(a)$ attaining the max for all a

2. for each i, $g_i^*(a_i^*) = a_i^*$ (stationarity)
3. $\int_0^1 g_i^*(a_i^*)\, di = 0$ (asset market clearing)

Characterization of Definition 7

The functional Euler equation (FOC + Envelope Theorem), for all i:

$$qu'(a + y_i - qg_i(a)) = \beta u'(g_i(a) + y_i - qg_i(g_i(a)))$$

Evaluating at $a = a_i^*$ with stationarity ($g_i^*(a_i^*) = a_i^*$):

$$q = \beta$$

Policy: $g_i^*(a) = a$ for all i (permanent-income result in recursive form).

Value: $V_i^*(a) = u(a(1 - q) + y_i)/(1 - \beta)$.

The asset distribution $\{a_i^*\}$ is indeterminate subject to $\int_0^1 a_i^* \, di = 0$. This model has no predictions for long-run wealth distributions (Chapter 21 addresses this).

Definition 8 (Steady-State CE, NGM)

A recursive steady-state CE consists of scalars r and w and a set of functions $V_i^*(k)$ and $g_i^*(k)$, and capital holdings k_i^*, for each $i \in [0,1]$, such that

1. for each i, $V_i^*(k)$ solves

   $$V_i(k) = \max_{k'} \left\{ u((1-\delta+r)k + we_i - k') + \beta V_i(k') \right\} \quad \forall k$$

   with $g_i^*(k)$ attaining the max for all k
2. $(K^*, 1)$ solves

   $$\max_{K,L} F(K,L) - rK - wL, \text{ where } \int_0^1 k_i^* \, di = K^*$$
3. for each i, $g_i^*(k_i^*) = k_i^*$ (stationarity)

Characterization of Definition 8

From the Euler equation at stationarity:

$$\beta = 1/(1-\delta+r),$$

which pins down r.

Then $r = F_1(K^*, 1)$ determines K^*, and $w = F_2(K^*, 1)$ determines w.

Policies: $g_i^*(k) = k$ and

$$V_i^*(k) = u(k(r-\delta) + we_i)/(1-\beta)$$

The distribution of capital is indeterminate, subject to

$$\int_0^1 k_i^* \, di = K^*.$$

Dynamic Recursive CE: Setup

Out of steady state: prices depend on aggregate capital K (relevant and predetermined).

Agent takes as given: price functions $r(K)$, $w(K)$, and a law of motion $K' = G(K)$.

Individual Bellman equation (must allow $k \neq K$):

$$V(k, K) = \max_{k'} \left\{ u((1 - \delta + r(K))k + w(K) - k') + \beta V(k', G(K)) \right\}$$

for all (k, K). Agent tracks K because prices depend on it.

Definition 9 (Recursive CE, NGM)

A recursive CE consists of functions $r(K)$, $w(K)$, $G^*(K)$, $V^*(k,K)$, and $g^*(k,K)$ such that

1. $V^*(k,K)$ solves

   $$V(k,K) = \max_{k'} \left\{ u((1-\delta+r(K))k + w(K) - k') + \beta V(k', G^*(K)) \right\}$$

   for all (k,K), with $g^*(k,K)$ attaining the max
2. $r(K) = F_1(K,1)$ and $w(K) = F_2(K,1)$ for all K
3. $G^*(K) = g^*(K,K)$ for all K (consistency)

Consistency: aggregate capital evolves according to individual choices when each agent holds $k = K$. The resource constraint is automatically satisfied.

Functional Euler Equation and Equivalence

FOC + Envelope Theorem, evaluated at $k = K$:

$$u'(F(K,1) + (1-\delta)K - G(K)) = \\ \beta u'(F(G(K),1) + (1-\delta)G(K) - G(G(K))) \cdot [1-\delta + F_1(G(K),1)]$$

This functional equation in G coincides with the planner's functional Euler equation from Chapter 4. Must be solved numerically unless $u = \log$, F Cobb-Douglas, $\delta = 1$.

Aggregation

With power utility, individual saving takes the form:

$$g(k, K) = \mu(K) + \lambda(K)k$$

- Saving is linear in own capital; slope $\lambda(K)$ is independent of k
- Wealth distribution does not affect aggregate saving $\Rightarrow$ not restrictive to use a representative agent
- With heterogeneous e_i: $g_i(k, K) = \mu_i(K) + \lambda(K)k$ (intercept varies, slope does not)

Definition 10 (Recursive CE with Valued Leisure)

A recursive CE consists of $r(K)$, $w(K)$, $G^*(K)$, $H^*(K)$, $V^*(k,K)$, $g^*(k,K)$, $h^*(k,K)$ such that

1. $V^*(k,K)$ solves

$$V(k,K) = \max_{k',\ell} \left\{ u((1-\delta+r(K))k + w(K)\ell - k') - v(\ell) + \beta V(k', G^*(K)) \right\}$$

   for all (k,K), with g^*, h^* attaining the max

2. $r(K) = F_1(K, H^*(K))$ and $w(K) = F_2(K, H^*(K))$ for all K
3. $G^*(K) = g^*(K,K)$ and $H^*(K) = h^*(K,K)$ for all K

The agent does not use H^* directly; G^* suffices because r and w capture how labor input changes with K. Aggregation breaks unless $k_{i,0}/e_i$ is equalized across agents.

Overlapping Generations

OLG: Setup

- Agents live two periods ($y =$ young, $o =$ old); no altruism, no bequests
- Cohort t utility: $u_t(c_y, c_o) = u(c_y) + \beta u(c_o)$
- Initial old ($t = -1$): $u_{-1}(c) = u(c)$
- Endowments: $(\omega_{y,t}, \omega_{o,t+1})$ for cohort t

Key differences from dynastic models:

- Simpler maximization (finite horizon)
- First-order dynamics (not second-order)
- Equilibria can be Pareto inefficient
- Negative real interest rates possible

Definition 11 (Sequential CE, OLG Endowment)

A sequential CE is $\{c^*_{i,t}\}_{t=0}^{\infty}$ for each $i \in \{y, o\}$, $\{a^*_{t+1}\}_{t=0}^{\infty}$, and $\{q_t\}_{t=0}^{\infty}$ such that

1. for each $t > 0$, $(c^*_{y,t}, c^*_{o,t+1}, a^*_{t+1})$ solves

$$\max_{c_y, c_o, a'} u(c_y) + \beta u(c_o)$$

   s.t. $c_y + q_t a' = \omega_{y,t}$ and $c_o = \omega_{o,t+1} + a'$
   and $c^*_{o,0} = \omega_{o,0}$
2. for all $t \geq 0$, $c^*_{y,t} + c^*_{o,t} = \omega_{y,t} + \omega_{o,t}$

Goods market clearing ⇔ $a^*_{t+1} = 0$ for all t: any saving by the young requires someone on the other side, but the old are in their last period and the next cohort is unborn.

Characterization of Definition 11

Since $a^*_{t+1} = 0$: **autarky**, $(c^*_{y,t}, c^*_{o,t}) = (\omega_{y,t}, \omega_{o,t})$.

Equilibrium price from the Euler equation at autarky (CRRA utility):

$$q_t = \beta \left(\frac{\omega_{y,t}}{\omega_{o,t+1}} \right)^{\sigma}$$

If $\omega_y > \omega_o$ (income declines over the life cycle): $q > 1$ possible $\Rightarrow$ **negative real interest rate**. Cannot happen in the dynastic model (unless aggregate endowments decline).

Intuition: No smoothing feasible (old cannot trade with unborn). Transferring from young to old (e.g. PAYG pension) can make all generations better off $\Rightarrow$ market outcome is inefficient. Chapter 6 analyzes this.

Definition 12 (Sequential CE, OLG Growth Economy)

Labor productivities e_y, e_o with $e_y + e_o = 1$; stationary $F(k, \ell)$.

A sequential CE is $\{c^*_{i,t}\}_{t=0}^{\infty}$ for each $i \in \{y, o\}$, $\{k^*_{t+1}\}_{t=0}^{\infty}$, $\{r_t\}_{t=0}^{\infty}$, $\{w_t\}_{t=0}^{\infty}$ such that

1. for each $t > 0$, $(c^*_{y,t}, c^*_{o,t+1}, k^*_{t+1})$ solves

$$\max_{c_y, c_o, k'} u(c_y) + \beta u(c_o)$$

   s.t. $c_y + k' = w_t e_y$ and $c_o = w_{t+1} e_o + (1-\delta+r_{t+1})k'$
   and $c^*_{o,0} = w_0 e_o + (1-\delta+r_0)k_0$
2. for all t, $r_t = F_1(k^*_t, 1)$ and $w_t = F_2(k^*_t, 1)$, with $k^*_0 \equiv k_0$
3. for all $t \geq 0$, $c^*_{y,t} + c^*_{o,t} + k^*_{t+1} = F(k^*_t, 1) + (1-\delta)k^*_t$

OLG Euler Equation and Dynamics

Substituting prices as functions of capital:

$$\begin{aligned} &u'(e_y F_2(k_t, 1) - k_{t+1}) = \\ &\quad \beta u'(e_o F_2(k_{t+1}, 1) + (1{-}\delta{+}F_1(k_{t+1}, 1))k_{t+1}) \cdot (1{-}\delta{+}F_1(k_{t+1}, 1)) \end{aligned}$$

- **First-order** difference equation: given k_t, solve for k_{t+1} and iterate forward
- Dynastic model: second-order (needs two boundary conditions)
- In general, $\beta(1 - \delta + F_1(\bar{k}, 1)) \neq 1$: the interest rate depends on life-cycle income
- No immediate connection to a planner's problem ⇒ uniqueness not guaranteed

OLG: Example

With $u = \log$, $e_y = 1$, $e_o = 0$, $F = K^\alpha L^{1-\alpha}$, $\delta = 1$:

$$k_{t+1} = \frac{\beta}{1+\beta}(1-\alpha)(k_t)^\alpha$$

- Log-linear dynamics, monotonic convergence to steady state
- Steady-state gross interest rate: $\alpha(1+\beta)/(1-\alpha)$, which can be < 1 if α is low enough

OLG Extensions

Perpetual youth: All individuals face a constant probability of death each period. As survival probability $\to 1$: approaches the dynastic model.

Warm glow: $u(c_y) + \beta u(c_o) + \varphi(b')$, where φ is utility from the act of giving (behavioral model).

Pure altruism: Parent values child's utility as the child does:

$$\varphi(b) = \max_{c_y, c_o, a', b'} \left\{ u(c_y) + \beta u(c_o) + \tilde{\beta}\varphi(b') \right\}$$

subject to budget constraints. Recursive structure $\Rightarrow$ dynastic life-cycle model with long-run interest rate $= 1/\tilde{\beta}$.

Summary

Key Takeaways (I)

1. **Three formulations**: Arrow-Debreu (Defs 1–4), sequential (Defs 5–6, 11–12), recursive (Defs 7–10). All yield the same allocations in dynastic economies.
2. **First Welfare Theorem**: With identical agents and no frictions, CE $=$ planner's solution.
3. **Aggregation**: Power utility $\Rightarrow g(k, K) = \mu(K) + \lambda(K)k \Rightarrow$ wealth distribution irrelevant. Breaks with valued leisure.
4. **Investment optimality**: $p_t = p_{t+1}[r_{t+1} + 1 - \delta]$.

Key Takeaways (II)

5. **OLG**: First-order dynamics; interest rate depends on life-cycle income; negative rates possible; can be Pareto inefficient.
6. **Pure altruism**: OLG → dynasty as special case; long-run $r = 1/\tilde{\beta}$.
7. **Departures from perfect competition** (frictions, monopoly, incomplete markets) $\Rightarrow$ CE $\neq$ planner's solution. Chapters 6 and beyond.