Chapter 6 Welfare

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Outline

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2. Tracing Out the Pareto Frontier
3. Inefficient Market Outcomes
4. Overlapping Generations: Efficiency
5. Optimal Government Policy
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The First Welfare Theorem

Motivation

Chapter 5 showed examples where the competitive equilibrium coincides with the planner's solution. The **First Welfare Theorem** (FWT) establishes this as a general result.

Under conditions of the FWT, decentralized decisions by self-interested agents, coordinated only through prices, produce a Pareto-optimal outcome. This is Adam Smith's invisible hand.

Pareto optimality is a minimal welfare criterion: silent on distribution. An allocation with extreme inequality can still be Pareto optimal.

When the FWT fails—due to taxes, externalities, missing markets, or market power—understanding the nature of the failure is essential for evaluating policy.

Abstract Setup

Finite set I of consumers, finite set J of firms. Vectors of goods x, y, ω.

Resource constraint:

$$\sum_{i\in I} x_i \leq \sum_{j\in J} y_j + \sum_{i\in I} \omega_i$$

- X_i: consumption possibility set for consumer i
- Y_j: production possibility set for firm j
- $\succsim_i$: preference ordering for consumer i
- $\theta_{i,j}$: consumer i's ownership share of firm j, with $\sum_i \theta_{i,j} = 1$ for all j
- p: price vector

Key assumption: Local non-satiation (LNS)—each consumer can always be made better off by an infinitesimal change in consumption.

Competitive Equilibrium (Abstract)

A competitive equilibrium is a consumption allocation $\{x_i^*\}_{\forall i}$, a production allocation $\{y_j^*\}_{\forall j}$, and a price system p^* such that

1. for each $i \in I$, x_i^* is in X_i and there is no $x \in X_i$ such that $x \succ_i x_i^*$ and

$$px \leq px_i^* = p\omega_i + \sum_j \theta_{i,j} py_j$$

2. for each $j \in J$, $y_j^* \in Y_j$ and there is no $y \in Y_j$ such that $py > py_j^*$
3. market clearing: the resource constraint holds with equality

Theorem 6.1 (First Welfare Theorem)

Theorem 6.1. An allocation that is part of a competitive equilibrium is Pareto optimal.

Proof (by contradiction). Suppose $\{\tilde{x}_i\}_{\forall i}$, $\{\tilde{y}_j\}_{\forall j}$ is feasible and Pareto dominates the equilibrium:

$$\forall i \in I : \tilde{x}_i \succsim_i x_i^*, \qquad \exists \tilde{i} \in I : \tilde{x}_{\tilde{i}} \succ_{\tilde{i}} x_{\tilde{i}}^*$$

From consumer optimization + LNS: $p\tilde{x}_i \geq px_i^*$ for all i, with strict inequality for $\tilde{i}$.

Summing household budgets:

$$\sum_{i \in I} p\tilde{x}_i > p\omega + \sum_{j \in J} py_j^*$$

From profit maximization: $p\tilde{y}_j \leq py_j^*$ for all j. Therefore:

$$\sum_{i \in I} p\tilde{x}_i > p\omega + \sum_{j \in J} p\tilde{y}_j$$

But feasibility requires $\sum_i p\tilde{x}_i \leq p\omega + \sum_j p\tilde{y}_j$. Contradiction. □

Mapping to Macroeconomic Models

Static model: One consumer with endowments $(k, 1)$, one firm with CRS production.

$$x \in \mathbb{R}^3_+, \quad y = (y, -k, -\ell) \in \mathbb{R}_+ \times \mathbb{R}^2_-, \quad \omega = (0, k, 1), \quad p = (1, r, w)$$

Dynamic model with $T < \infty$: Goods at different dates are additional market goods. Vectors become T times longer.

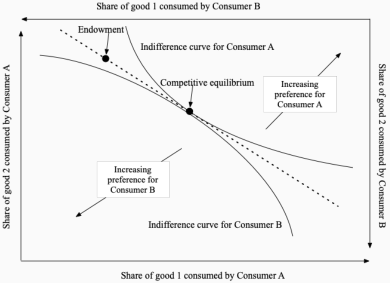
Infinite horizon: Vectors are infinite-dimensional. The proof carries through provided total expenditures are finite:

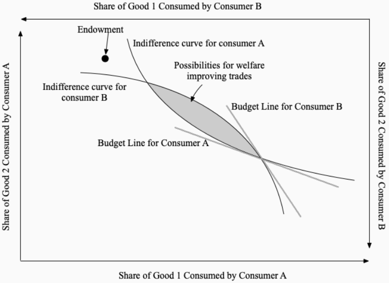
$$\sum_{t=0}^{\infty} p_t c_{i,t} < \infty$$

Typically satisfied in dynastic models (p_t falls at rate $\beta < 1$). Can fail in OLG models.

Intuition: Edgeworth Box

Endowment economy with two goods and two consumers





Panel (a) (efficient equilibrium): Both consumers face the same prices. Indifference curves tangent to the budget line and to each other $\Rightarrow$ no Pareto improvement possible.

Panel (b) (distorted market): Consumers face different (after-tax) prices. Indifference curves intersect $\Rightarrow$ shaded region represents Pareto-improving reallocations.

Tracing Out the Pareto Frontier

The Social Planner's Problem

A social planner maximizes a weighted sum of utilities:

$$\max_{\{c_{i,t}\}} \sum_{i \in I} \mu_i U_i, \qquad U_i = \sum_{t=0}^{\infty} \beta^t u(c_{i,t})$$

subject to the resource constraint $\sum_{i \in I} c_{i,t} \leq \sum_{i \in I} \omega_{i,t}$ for all t.

The weights μ_i are called **Negishi weights** (or Pareto weights).

Any solution to this problem must be Pareto optimal: if it were not, the planner could increase the objective.

Negishi Weights and Competitive Equilibrium

Planner's FOC:

$$c_{i,t} = (u')^{-1}\left(\beta^{-t}\frac{1}{\mu_i}\psi_t\right)$$

where ψ_t is the multiplier on the resource constraint at t.

Competitive equilibrium FOC (from Chapter 5):

$$c_{i,t} = (u')^{-1}\big(\beta^{-t}\lambda_i p_t\big)$$

Setting $\mu_i = 1/\lambda_i$ and $\psi_t = p_t$: the two allocations coincide.

Implication: The CE is optimal for a planner who assigns higher weight to agents with higher date-0 wealth (λ_i lower $\Rightarrow$ μ_i higher).

By varying $\{\mu_i\}$, we trace the **Pareto frontier**. The **Second Welfare Theorem** gives conditions (convexity) under which any Pareto-optimal allocation can be supported as a CE with appropriate lump-sum transfers.

Inefficient Market Outcomes

Taxes

Lump-sum taxes: Do not affect any marginal conditions ⇒ act as redistribution along the Pareto frontier. FWT holds.

Proportional labor tax τ_ℓ: With valued leisure, the after-tax wage is $w_t(1-\tau_\ell)$. MRS between consumption and leisure $\neq$ MRT ⇒ FWT fails.

- Exception: If leisure is not valued (inelastic labor), the tax is non-distortionary

Capital income tax τ_k: After-tax return $r_t(1-\tau_k)$ enters the Euler equation. MRS between t and $t+1$ goods $\neq$ MRT ⇒ FWT fails.

In both cases, consumers and firms face different effective prices ⇒ the step $\sum_i p\tilde{x}_i > p\omega + \sum_j py_j^*$ may not hold.

Externalities

An externality arises when one agent's activity has payoff relevance to others, not reflected in prices.

Example: Firm j produces $A(\bar{y})k_j^{\alpha}\ell_j^{1-\alpha}$, where A is decreasing in average output $\bar{y}$.

Equilibrium hours (log utility, labor supply with income/substitution cancellation):

$$1 - \alpha = B\ell^{1+1/\theta}$$

Efficient hours (planner internalizes the externality):

$$\frac{1-\alpha}{1 - A'(y)k^{\alpha}\ell^{1-\alpha}} = B\ell^{1+1/\theta}$$

The equilibrium is inefficient unless $A'(y) = 0$. The proof of the FWT fails because firms' production possibility sets Y_j are endogenous and interdependent.

Missing Markets: Borrowing Constraints

Two-agent dynastic endowment economy. Endowments alternate: agent 1 has $2\omega/3$ in odd periods, $\omega/3$ in even periods; agent 2 reversed.

Unrestricted equilibrium: Full consumption smoothing, $c_{1,t} = c_1$ and $c_{2,t} = c_2$ for all t. Gross interest rate $= 1/\beta$.

With borrowing constraint $a_{t+1} \geq 0$: Euler equation becomes an inequality:

$$q_t u'(c_t) \geq \beta u'(c_{t+1})$$

Result: autarky. No smoothing. Clearly not Pareto optimal.

Equilibrium bond price: $q_t = 2\beta$ (gross interest rate $= 1/(2\beta)$, can be negative if $\beta > 0.5$). The unconstrained agent sets the price; the constrained agent would like to borrow but cannot.

Lack of Commitment

When agents can default on debt without punishment: leads to de-facto borrowing constraints (rational lenders refuse to lend).

With punishment: intertemporal trade occurs, but the punishment mechanism itself must be committed to.

Examples studied later:

- Government cannot commit to future tax policy (Chapter 15)
- Sovereign debt default (Chapter 24)

Monopolistic Competition: Setup

Consumer with CES preferences over a continuum of goods:

$$U\big(\{c(i)\}_{i=0}^{1}, L\big) = u\left(\left(\int_0^1 c(i)^{1-\frac{1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}}\right) - v(L)$$

where $\varepsilon > 1$ is the elasticity of substitution.

Consumer budget:

$$\int_0^1 p(i)c(i)\, di = y$$

CES price index (unit cost of composite C):

$$P = \left(\int_0^1 p(i)^{1-\varepsilon}\, di\right)^{\frac{1}{1-\varepsilon}}$$

Demand for variety i:

$$c(i) = C\left(\frac{p(i)}{P}\right)^{-\varepsilon}$$

Monopolistic Competition: Equilibrium

Each firm produces $c(i) = A\ell(i)$ and sets price facing the demand curve. Total labor supply $L = 1$ (inelastic).

Firm's problem: $\max_{p,c,\ell} pc - w\ell$ subject to $c = A\ell$ and $p = P(c/C)^{-1/\varepsilon}$.

Optimal price (constant markup $\mu = \varepsilon/(\varepsilon - 1) > 1$):

$$p = \frac{\varepsilon}{\varepsilon - 1} \cdot \frac{w}{A}$$

Equilibrium wage: $w = A(1 - 1/\varepsilon)$ (below marginal product).

Equilibrium output (with $u = \log$, $P = 1$):

$$A(1 - 1/\varepsilon) = Cv'(C/A)$$

Planner's optimum: $A = Cv'(C/A)$. Output and hours are **too low** in equilibrium.

Quantifying Welfare Losses

Compare two allocations: (c_l, ℓ_l) under lump-sum taxes vs. (c_d, ℓ_d) under distortionary taxes. Define the **consumption-equivalent welfare loss** $\Delta > 0$:

$$u(c_l(1-\Delta), \ell_l) = u(c_d, \ell_d)$$

Δ is the percentage reduction in consumption (at fixed hours) that makes the better allocation as good as the worse one.

- Has a real interpretation (unlike "utils")
- In dynamic models: Δ applies uniformly across all periods
- Alternative: Δ as a percentage *increase* in the worse allocation (gives a different number)

Overlapping Generations: Efficiency

OLG Endowment Economy: Setup

Representative consumer per cohort, two-period lives, $u_t(c_y, c_o) = \log c_y + \log c_o$. Stationary endowments (ω_y, ω_o).

Competitive equilibrium: sequential trading, no borrowing constraints.

Consumer born at t solves:

$$\max_{c_y, c_o} \log c_y + \log c_o \quad \text{s.t.} \quad c_y + q_t c_o = \omega_y + q_t \omega_o$$

Equilibrium: autarky, $c_{y,t} = \omega_y$, $c_{o,t} = \omega_o$, bond price $q_t = \omega_y / \omega_o$.

OLG Endowment Economy: Is the Equilibrium Efficient?

Case 1: $\omega_y = 3$, $\omega_o = 1$. Bond price $q = 3$ (gross interest rate $1/3$).

Alternative allocation: $\tilde{c}_{y,t} = \tilde{c}_{o,t} = 2$ for all t (transfer 1 from young to old each period).

- Initial old: strictly better $(2 > 1)$
- All subsequent generations: $\log 2 + \log 2 = \log 4 > \log 3 + \log 1 = \log 3 \Rightarrow$ strictly better

$\Rightarrow$ Equilibrium is **not** Pareto optimal.

Case 2: $\omega_y = 1$, $\omega_o = 3$. Bond price $q = 1/3$ (gross interest rate 3).

The $(2, 2)$ alternative improves all generations born at $t \geq 0$ but makes the initial old **worse off** $(2 < 3)$. No other allocation can Pareto dominate $\Rightarrow$ equilibrium **is** Pareto optimal.

Why the FWT Can Fail in OLG

In the proof of the FWT, we sum budget constraints over all agents. With OLG:

- I is infinite (infinitely many cohorts)
- Present value of total expenditures can be infinite

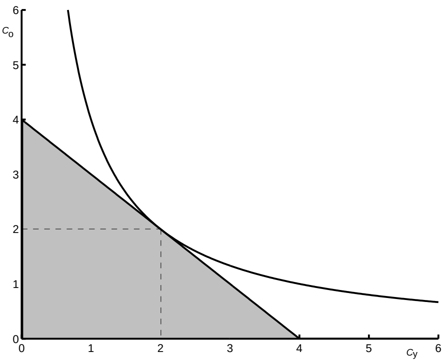
$q_t = 3$ for all t: $\sum_{t=0}^{\infty} 1/p_t = \sum_{t=0}^{\infty} 3^{-t} = 3/2$ (finite) $\Rightarrow$ proof fails.

$q_t = 1/3$ for all t: $\sum_{t=0}^{\infty} 1/p_t = \sum_{t=0}^{\infty} 3^{t} = \infty \Rightarrow$ proof goes through.

The combination of (i) infinite time and (ii) infinitely many finitely-lived consumers can cause markets to fail, even without any frictions.

Pareto Optimality and Intergenerational Transfers

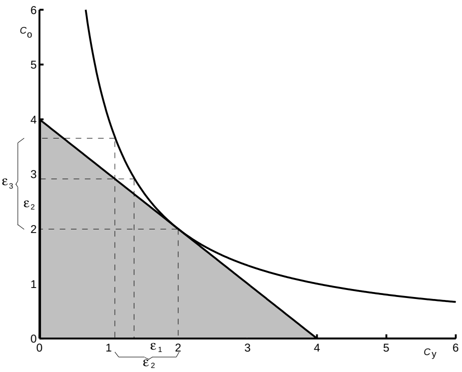
Pareto optimality of the $(2, 2)$ allocation



No stationary allocation Pareto dominates $(2, 2)$. What about non-stationary allocations?

Pareto Optimality and Intergenerational Transfers

$(2, 2)$ cannot be improved upon



Transfer ε_1 from young to old at t requires compensation $\varepsilon_2 > \varepsilon_1$ at $t + 1$ (convexity of indifference curves). Required transfers grow without bound $\Rightarrow$ infeasible.

The Balasko-Shell Theorem (Endowment Economy)

Theorem (Balasko and Shell, 1980): A competitive equilibrium in an OLG endowment economy is Pareto optimal if and only if

$$\sum_{t=0}^{\infty} \frac{1}{p_t} = \infty$$

where $p_t = \prod_{\tau=0}^{t-1} q_\tau$ is the Arrow-Debreu price of consumption at t.

Implication: When the gross real interest rate converges, the equilibrium is efficient if and only if the **limit net real interest rate is non-negative**.

With growth at rate g: efficient if and only if the limit interest rate $\geq g$.

When the equilibrium is inefficient, a PAYG transfer scheme (young → old) makes all generations better off. But this requires government commitment to an infinite sequence of transfers.

Dynamic Efficiency with Production

Neoclassical production economy with OLG. Agent's budget:

$$c_y + s = \omega_y w_t, \qquad c_o = s(1-\delta+r_{t+1}) + \omega_o w_{t+1}$$

Asset market clearing: $s_t = k_{t+1}$. Competitive pricing: $r_t = F_1(k_t, \omega_y+\omega_o)$, $w_t = F_2(k_t, \omega_y+\omega_o)$.

Theorem 6.2. Define $R_t = 1-\delta+F_1(k_t,\omega_t)$. Then $\{k_t\}_{t=0}^{\infty}$ is dynamically efficient if and only if

$$\sum_{t=0}^{\infty}\prod_{s=1}^{t} R_s(k_s) = \infty$$

Steady state: efficient iff $R = F_1(k_{ss},\omega)+1-\delta \geq 1$, i.e., the net interest rate is non-negative.

Dynamic Inefficiency: Intuition

When $R < 1$ at steady state: Reduce saving by ε at t. Resources freed at t. At $t+1$, production falls by less than ε (since marginal return < 1). Reduce saving by ε again at $t+1$: net resources freed. Repeat forever ⇒ **free lunch** at all dates.

When $R \geq 1$: Reducing saving by ε at t reduces resources at $t+1$ by *more* than ε. Future reductions grow and eventually become infeasible.

Oversaving: In OLG models, equilibrium savings can be "too high" because each generation saves for retirement without regard for the aggregate capital stock. Fundamentally different from the dynastic model, where the TVC prevents over-accumulation.

The warm-glow model (Chapter 5) can also generate dynamic inefficiency: when $v(b) = Ab$ (linear), convergence to a steady state with $r - \delta < 0$.

Optimal Government Policy

Correcting Market Failures

Externalities (Pigou, 1920): A tax equal to the marginal external damage, evaluated at the efficient allocation, restores optimality. An alternative: assign property rights (Coase) and let the market internalize the externality. Not always feasible.

Monopoly: Subsidize production at a per-unit rate to counteract under-production. Alternatively, anti-trust regulation. But monopoly power can incentivize innovation through patents $\Rightarrow$ trade-off (Chapter 13).

Distortionary taxes: Compare alternative feasible tax systems (Ramsey analysis, Chapter 15). Which tax base is least distortionary?

Missing markets: Beware the “chicken model”—if a market is missing, it is usually missing for a reason (moral hazard, adverse selection). Mirrlees (1971) approach: model the friction explicitly.

Redistribution

A separate aim from efficiency: achieving a more equitable distribution, even if some agents are made worse off.

Common approach: **utilitarian social welfare function** with equal weights:

$$\max \sum_i \pi_i u(c_i) \quad \text{s.t.} \quad \sum_i \pi_i c_i = \sum_i \pi_i \omega_i$$

where π_i is the fraction of type-i agents.

With non-distortionary redistribution: equal weights ⇒ equal consumption. With distortionary taxes: optimal redistribution is partial.

Behind the veil of ignorance: Before knowing one's type, an agent would choose the utilitarian optimum as an insurance scheme. Cannot be offered by markets (agents not yet born), but government can implement it.

Summary

Key Takeaways (I)

1. **First Welfare Theorem**: Under LNS, a competitive equilibrium is Pareto optimal. Proof: the alternative allocation must be more expensive (consumer optimization) but cannot be (profit maximization + feasibility). Contradiction.
2. **Negishi weights**: CE is optimal for a planner with $\mu_i = 1/\lambda_i$. Varying weights traces the Pareto frontier. Second Welfare Theorem: any Pareto optimum can be decentralized with lump-sum transfers (under convexity).
3. **Market failures**: Distortionary taxes (consumers and firms face different prices), externalities (possibility sets interdependent), missing markets (additional constraints, pecuniary externalities), lack of commitment, monopoly power (price-setting, not price-taking).

Key Takeaways (II)

4. **OLG efficiency**: Even without frictions, OLG equilibria can be Pareto inefficient. Balasko-Shell: efficient iff $\sum 1/p_t = \infty$, i.e., the asymptotic net interest rate ≥ 0 (or $\geq g$ with growth). Dynamic inefficiency = oversaving.
5. **Welfare measurement**: Consumption-equivalent Δ — the percentage reduction in consumption that equates utility across allocations.
6. **CES and monopolistic competition**: Markup $\mu = \varepsilon/(\varepsilon - 1)$; CES price index $P = (\int p(i)^{1-\varepsilon} di)^{1/(1-\varepsilon)}$; equilibrium under-produces relative to planner.
7. **Policy**: Pigou taxes for externalities, Ramsey analysis for tax design, Mirrlees for informational frictions. Beware the "chicken model."