

Chapter 9: Continuous-Time Analytical Techniques

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Outline

1. Introduction
2. Basic Tools and Notation
3. Optimization: The Maximum Principle
4. Continuous-Time Growth Models
5. Uncertainty: The Poisson Process
6. Summary

Introduction

Why Continuous Time?

Most of this textbook uses discrete time. Discrete time is:

- Intuitive and aligns well with how data are measured
- Convenient for computation and numerical methods

But continuous time offers particular advantages:

- Certain problems become **mathematically more tractable**
- Key relationships involving differential equations derived more transparently with “paper and pencil” techniques
- Especially helpful when variables evolve **smoothly** over time

This chapter introduces the **core analytical techniques** specific to continuous-time models.

Discrete vs. Continuous Time: Intuition

Discrete time: Track variables at specific intervals (monthly, quarterly, yearly)

- Change is modeled from one period to the next
- Dynamics described by **difference equations**
- Solutions are **sequences** of values at discrete points

Continuous time: Time is a continuous flow

- Variables evolve smoothly at every instant
- Dynamics described by **differential equations**
- Solutions are **continuous functions** of time
- More precise depiction of gradual economic adjustments

Basic Tools and Notation

Notation

Discrete time: subscripts X_t for $t = 0, 1, 2, \dots$

Continuous time: functions of time $X(t)$ where $t \in \mathbb{R}_+$

Key mathematical concept: the **time derivative**

$$\frac{dX(t)}{dt} \equiv \dot{X}(t)$$

“Dot notation” (Newton notation): $\dot{X}(t)$ measures the **instantaneous rate of change**.

Growth Rates: Discrete vs. Continuous

Discrete time: If X_t grows at constant rate γ :

$$\frac{X_{t+1} - X_t}{X_t} = \gamma$$

Solution (difference equation):

$$X_t = (1 + \gamma)^t X_0$$

Continuous time: If instantaneous growth rate is γ :

$$\frac{\dot{X}(t)}{X(t)} = \gamma$$

Solution (differential equation):

$$X(t) = e^{\gamma t} X(0)$$

Verification: Differentiate: $\dot{X}(t) = \gamma e^{\gamma t} X(0) = \gamma X(t)$

The Log-Derivative Trick

A particularly useful property in continuous time:

$$\frac{\dot{X}(t)}{X(t)} = \frac{d}{dt} \log(X(t))$$

Interpretation: The derivative of the log of a variable gives its **instantaneous growth rate**.

Implication: If a variable grows at a constant rate, its logarithm traces a **straight line** over time. This is why economists plot time series on a log scale.

Application: Growth Accounting

Consider the Cobb-Douglas production function:

$$Y(t) = z(t) K(t)^\alpha L(t)^{1-\alpha}$$

Taking logs:

$$\log Y(t) = \log z(t) + \alpha \log K(t) + (1 - \alpha) \log L(t)$$

Taking the time derivative and applying the log-derivative trick:

$$\frac{\dot{Y}(t)}{Y(t)} = \frac{\dot{z}(t)}{z(t)} + \alpha \frac{\dot{K}(t)}{K(t)} + (1 - \alpha) \frac{\dot{L}(t)}{L(t)}$$

This is the growth accounting equation. The continuous-time derivation is particularly clean and transparent.

More generally: The log-derivative trick is powerful for products, quotients, and powers—all common in macroeconomics.

Connecting Discrete and Continuous Time

Discrete: $X_t = (1 + \gamma)^t X_0$. Taking logs and differences:

$$\log X_{t+1} - \log X_t = \log(1 + \gamma) \approx \gamma \quad (\text{for small } \gamma)$$

Continuous: $X(t) = e^{\gamma t} X(0)$. Taking logs and differentiating:

$$\frac{d}{dt} \log X(t) = \gamma \quad (\text{exact})$$

The key connection: $\log(1 + \gamma) \approx \gamma$ for small γ , so discrete-time log-differences approximate continuous-time growth rates.

The exponential $e^{\gamma t}$ in continuous time plays the role of $(1 + \gamma)^t$ in discrete time.

Optimization: The Maximum Principle

Finite-Horizon Consumption-Saving Problem (Discrete Time)

Consumer solves:

$$\max_{\{c_t, a_{t+1}\}_{t=0}^T} \sum_{t=0}^T \beta^t u(c_t)$$

subject to

$$a_{t+1} = w + (1 + r)a_t - c_t, \quad a_0 \text{ given}$$

$$a_{T+1} \geq 0$$

- $\beta \in (0, 1)$: discount factor; $u(\cdot)$: strictly increasing, strictly concave
- c_t : consumption; a_t : assets; w, r : exogenous
- **Control variable**: c_t (chosen at t)
- **State variable**: a_t (carries over from previous choices)

The Discrete-Time Hamiltonian

Rewrite the budget constraint in difference form:

$$a_{t+1} - a_t = ra_t + w - c_t$$

Define the **discrete-time Hamiltonian**:

$$H_t \equiv \beta^t u(c_t) + \mu_t(ra_t + w - c_t)$$

where μ_t is the costate variable (Lagrange multiplier).

The Lagrangian becomes:

$$L = \sum_{t=0}^T H_t - \sum_{t=0}^T \mu_t(a_{t+1} - a_t) + \lambda a_{T+1}$$

First-Order Conditions (Discrete Time)

FOC for control c_t ($t = 0, 1, \dots, T$):

$$\frac{\partial H_t}{\partial c_t} = 0$$

FOC for state a_{t+1} ($t = 0, 1, \dots, T - 1$):

$$\frac{\partial H_{t+1}}{\partial a_{t+1}} + \mu_{t+1} - \mu_t = 0$$

Note the extra term $\mu_{t+1} - \mu_t$: the costate variable can change over time, and this must be accounted for.

Terminal condition: $-\mu_T + \lambda = 0$

Kuhn-Tucker: $\lambda \geq 0$, $a_{T+1} \geq 0$, $\lambda a_{T+1} = 0$

Euler Equation and Transversality Condition (Discrete Time)

From the control FOC: $\beta^t u'(c_t) = \mu_t$

From the state FOC: $\mu_t = (1 + r)\mu_{t+1}$

Combining:

$$u'(c_t) = \beta(1 + r) u'(c_{t+1})$$

This is the standard **discrete-time Euler equation**.

Transversality condition (combining terminal and KT conditions):

$$\beta^T u'(c_T) a_{T+1} = 0$$

Equivalently: $a_{T+1}/(1 + r)^T = 0$.

In the finite-horizon case: $a_{T+1} = 0$ (consume everything by the end).

Infinite Horizon (Discrete Time)

For $T \rightarrow \infty$: same Euler equation, with the **no-Ponzi game condition**:

$$\lim_{T \rightarrow \infty} \frac{a_{T+1}}{(1+r)^T} \geq 0$$

and the **transversality condition** becomes:

$$\lim_{T \rightarrow \infty} \beta^T u'(c_T) a_{T+1} = 0$$

Note: The nPg condition $\lim_{T \rightarrow \infty} a_{T+1}/(1+r)^T \geq 0$ is *not* the same as $\lim_{T \rightarrow \infty} a_{T+1} \geq 0$. It restricts the *present value* of terminal assets. (Consider $a_t = a > 0$ for all t : satisfies the former but not the latter.)

The Continuous-Time Problem

$$\max_{c(t), a(t)} \int_0^{\infty} e^{-\rho t} u(c(t)) dt$$

subject to

$$\dot{a}(t) = w + r a(t) - c(t), \quad a(0) \text{ given}$$

$$\lim_{T \rightarrow \infty} e^{-rT} a(T) \geq 0$$

Key differences from discrete time:

- Sum \rightarrow integral: $\sum_{t=0}^{\infty} \beta^t u(c_t) \rightarrow \int_0^{\infty} e^{-\rho t} u(c(t)) dt$
- Difference \rightarrow derivative: $a_{t+1} - a_t \rightarrow \dot{a}(t)$
- Discount factor \rightarrow discount rate: $\beta^t \rightarrow e^{-\rho t}$
- No-Ponzi game: $1/(1+r)^T \rightarrow e^{-rT}$

Note: Connecting β and ρ

In discrete time: discount factor $\beta = 1/(1 + \rho)$.

Divide one period into subperiods of length Δ , discounting each by rate $\rho\Delta$. Discounting t periods ($= t/\Delta$ subperiods):

$$f(\Delta, t) = \left(\frac{1}{1 + \rho\Delta} \right)^{t/\Delta}$$

Taking $\Delta \rightarrow 0$:

$$\lim_{\Delta \rightarrow 0} f(\Delta, t) = \lim_{\Delta \rightarrow 0} \left[(1 + \rho\Delta)^{1/(\rho\Delta)} \right]^{-\rho t} = e^{-\rho t}$$

using $e = \lim_{x \rightarrow 0} (1 + x)^{1/x}$.

\Rightarrow The continuous-time $e^{-\rho t}$ is the limit of discrete-time $\beta^t = [1/(1 + \rho)]^t$ as the period length goes to zero. Larger $\rho \Leftrightarrow$ smaller $\beta \Leftrightarrow$ less patient.

The Present-Value Hamiltonian

Define the **present-value Hamiltonian**:

$$H(t) \equiv e^{-\rho t} u(c(t)) + \mu(t)(ra(t) + w - c(t))$$

where $\mu(t)$ is the costate variable.

First-order conditions:

Control $c(t)$:

$$\frac{\partial H(t)}{\partial c(t)} = 0$$

State $a(t)$:

$$\frac{\partial H(t)}{\partial a(t)} + \dot{\mu}(t) = 0$$

Transversality condition:

$$\lim_{T \rightarrow \infty} e^{-\rho T} u'(c(T)) a(T) = 0$$

Differences from Discrete Time

Three key differences in the continuous-time FOCs:

1. Change in μ is $\dot{\mu}(t)$ instead of $\mu_{t+1} - \mu_t$
2. The relevant derivative is $\partial H(t)/\partial a(t)$ instead of $\partial H_{t+1}/\partial a_{t+1}$
 - No “next period” in continuous time
3. Discounting in TVC is $e^{-\rho T}$ instead of β^T

Heuristic derivation: Construct Lagrangian

$$L = \int_0^{\infty} H(t)dt - \int_0^{\infty} \mu(t)\dot{a}(t)dt$$

Apply integration by parts to the second term:

$$L = \int_0^{\infty} H(t)dt - \left[\lim_{T \rightarrow \infty} \mu(T)a(T) - \mu(0)a(0) \right] + \int_0^{\infty} \dot{\mu}(t)a(t)dt$$

Taking FOCs on $c(t)$ and $a(t)$ yields the above conditions.

Present-Value vs. Current-Value Hamiltonians

The **present-value Hamiltonian** includes the discount factor $e^{-\rho t}$.

The **current-value Hamiltonian** evaluates utility at current value:

$$\hat{H}(t) \equiv u(c(t)) + \hat{\mu}(t)(ra(t) + w - c(t))$$

Modified FOCs:

$$\frac{\partial \hat{H}(t)}{\partial c(t)} = 0$$

$$\frac{\partial \hat{H}(t)}{\partial a(t)} + \dot{\hat{\mu}}(t) - \rho \hat{\mu}(t) = 0$$

Relationship: $\mu(t) = e^{-\rho t} \hat{\mu}(t)$

The extra $-\rho \hat{\mu}(t)$ term accounts for discounting that was removed from the Hamiltonian.

Deriving the Continuous-Time Euler Equation

From the control FOC:

$$e^{-\rho t} u'(c(t)) = \mu(t)$$

From the state FOC:

$$r\mu(t) + \dot{\mu}(t) = 0 \quad \Rightarrow \quad \frac{\dot{\mu}(t)}{\mu(t)} = -r$$

Apply the log-derivative trick to $e^{-\rho t} u'(c(t)) = \mu(t)$:

$$-\rho + \frac{u''(c(t))}{u'(c(t))} \dot{c}(t) = \frac{\dot{\mu}(t)}{\mu(t)} = -r$$

Rearranging:

$$-\frac{u''(c(t))c(t)}{u'(c(t))} \frac{\dot{c}(t)}{c(t)} = r - \rho$$

This is the **continuous-time Euler equation**.

Interpreting the Continuous-Time Euler Equation

$$-\frac{u''(c(t))c(t)}{u'(c(t))} \frac{\dot{c}(t)}{c(t)} = r - \rho$$

- $-u''(c)c/u'(c)$: **coefficient of relative risk aversion**
- With CRRA utility $u(c) = (c^{1-\sigma} - 1)/(1 - \sigma)$: coefficient = σ
- \dot{c}/c : consumption growth rate
- r : benefit of saving (interest rate)
- ρ : desire to consume sooner (discount rate)

Implications: Consumption growth = $(r - \rho)/\sigma$

- ρ large (impatient) \Rightarrow low or negative consumption growth
- ρ small (patient) \Rightarrow faster consumption growth
- r large (high return to saving) \Rightarrow faster consumption growth

Connecting Discrete and Continuous Euler Equations

Discrete: $u'(c_t) = \beta(1+r)u'(c_{t+1})$

Continuous: $-\frac{u''(c(t))c(t)}{u'(c(t))} \cdot \frac{c'(t)}{c(t)} = r - \rho$

To see the connection, take a first-order Taylor expansion of $u'(c_{t+1})$:

$$u'(c_{t+1}) \approx u'(c_t) + u''(c_t)(c_{t+1} - c_t)$$

Substitute into the discrete Euler equation with $\beta = 1/(1+r)$:

$$-\frac{u''(c_t)c_t}{u'(c_t)} \cdot \frac{c_{t+1} - c_t}{c_t} = \frac{r - \rho}{1+r}$$

This closely parallels the continuous-time Euler equation. Despite different notations, **both capture the same trade-off** between impatience (ρ) and the rewards to saving (r).

Continuous-Time Growth Models

Revisiting Growth Models in Continuous Time

In Chapters 3 and 4: Solow model and neoclassical growth model in discrete time.

Purpose of this section:

1. Show how the same economic mechanisms are formulated using **differential equations**
2. Provide hands-on examples of the tools developed earlier
3. Continuous-time approach: more streamlined characterization of **transition dynamics** and **steady states**
4. Clarify connections and distinctions between discrete- and continuous-time techniques

Solow Model: Setup

Aggregate production function:

$$Y(t) = F(K(t), A(t)L(t))$$

Labor and technology grow at constant rates:

$$\frac{\dot{L}(t)}{L(t)} = n, \quad \frac{\dot{A}(t)}{A(t)} = \gamma$$

Capital accumulates via:

$$\dot{K}(t) = I(t) - \delta K(t)$$

Constant saving rate:

$$I(t) = sF(K(t), A(t)L(t)), \quad s \in (0, 1)$$

Solow Model: Capital Per Effective Worker

Define $\tilde{k}(t) \equiv K(t)/(A(t)L(t))$. Using the log-derivative trick:

$$\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} = \frac{\dot{K}(t)}{K(t)} - \frac{\dot{A}(t)}{A(t)} - \frac{\dot{L}(t)}{L(t)}$$

Substituting the laws of motion and using CRS ($F(K, AL)/AL = f(\tilde{k})$):

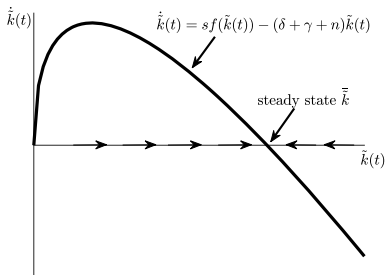
$$\frac{\dot{\tilde{k}}}{\tilde{k}} = \frac{sf(\tilde{k})}{\tilde{k}} - (\delta + \gamma + n)$$

Multiplying by \tilde{k} :

$$\dot{\tilde{k}}(t) = sf(\tilde{k}(t)) - (\delta + \gamma + n)\tilde{k}(t)$$

Solow Model: Phase Diagram and Convergence

Solow model in continuous time



- Curve: RHS of the law of motion. Crosses zero at \tilde{k} .
- $\tilde{k} < \tilde{k}$: $\dot{\tilde{k}} > 0 \Rightarrow$ capital per effective worker **increases**
- $\tilde{k} > \tilde{k}$: $\dot{\tilde{k}} < 0 \Rightarrow$ capital per effective worker **decreases**
- **Convergence**: regardless of $\tilde{k}(0)$, economy approaches \tilde{k}

Balanced growth path: at \tilde{k} , per-capita GDP $Y/L = f(\tilde{k})A(t)$ grows at rate γ .

Neoclassical Growth (Ramsey) Model: Planner's Problem

No population growth or technological progress (can be added).

The planner solves:

$$\max_{c(t), k(t)} \int_0^{\infty} e^{-\rho t} u(c(t)) dt$$

subject to

$$\dot{k}(t) = f(k(t)) - \delta k(t) - c(t)$$

and $c(t), k(t) \geq 0$ for all t , $k(0)$ given.

No need for explicit no-Ponzi game condition (closed economy: planner cannot borrow against the future).

Neoclassical Growth Model: Hamiltonian and FOCs

Hamiltonian:

$$H(t) = e^{-\rho t} u(c(t)) + \mu(t)(f(k(t)) - \delta k(t) - c(t))$$

FOCs:

$$\frac{\partial H}{\partial c(t)} = 0 \quad \Rightarrow \quad e^{-\rho t} u'(c(t)) = \mu(t)$$

$$\frac{\partial H}{\partial k(t)} + \dot{\mu}(t) = 0 \quad \Rightarrow \quad \mu(t)(f'(k(t)) - \delta) + \dot{\mu}(t) = 0$$

Transversality condition:

$$\lim_{T \rightarrow \infty} e^{-\rho T} u'(c(T)) k(T) = 0$$

Neoclassical Model: Euler Equation

With CRRA utility $u(c) = (c^{1-\sigma} - 1)/(1 - \sigma)$, following the same derivation steps:

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\sigma} (f'(k(t)) - (\delta + \rho))$$

The economy is governed by the system of **two differential equations**:

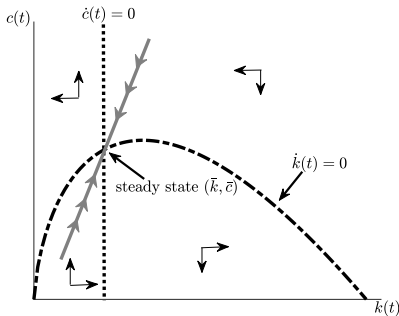
- Law of motion for capital: $\dot{k}(t) = f(k(t)) - \delta k(t) - c(t)$
- Euler equation for consumption:

$$\dot{c}(t)/c(t) = (f'(k(t)) - \delta - \rho)/\sigma$$

Together with initial condition $k(0)$ and the TVC, these determine the dynamic paths of $k(t)$ and $c(t)$.

Phase Diagram

Phase diagram for Ramsey model in continuous time



From the capital equation: $\dot{k}(t) = 0$ locus is the dash-dot curve
 $c(t) = f(k(t)) - \delta k(t)$

From the Euler equation: $\dot{c}(t) = 0$ locus is the vertical dotted
line at k^{**} where $f'(k^{**}) = \delta + \rho$

k^{**} : the **modified golden rule** capital stock.

Saddle Path and Convergence

- Starting from $k(0)$, the planner chooses $c(0)$ on the **saddle path** (solid gray line in the Figure)
- On the saddle path: $(k(t), c(t))$ gradually approaches the steady state (\bar{k}, \bar{c}) where $\bar{k} = k^{**}$
- This path satisfies the TVC because $k(t)$ and $c(t)$ become constant in the long run

Why other paths fail:

- $c(0)$ too high: $(k(t), c(t))$ eventually reaches $k(t) = 0$ — economy cannot produce, path infeasible
- $c(0)$ too low: $k(t)$ grows large, $c(t) \rightarrow 0$ — over-accumulation of capital, violates TVC

⇒ The Ramsey model exhibits **convergence** to the steady state.

Golden Rule vs. Modified Golden Rule

Golden rule: $f'(k^{GR}) = \delta$ — maximizes steady-state consumption
 $c = f(k) - \delta k$

Modified golden rule: $f'(k^{**}) = \delta + \rho$ — steady state of the Ramsey model

Because $\rho > 0$: $k^{**} < k^{GR}$

Intuition: Consumers' impatience ($\rho > 0$) leads to less capital accumulation in steady state than what would maximize long-run consumption. They prefer consuming more today rather than waiting for the Golden Rule level of capital.

Market Equilibrium in Continuous Time

Continuum of identical households $i \in [0, 1]$:

$$\max_{c_i(t), k_i(t)} \int_0^{\infty} e^{-\rho t} u(c_i(t)) dt$$

subject to

$$\dot{k}_i(t) = r(t) k_i(t) + w(t) - \delta k_i(t) - c_i(t)$$

Household Euler equation:

$$\frac{\dot{c}_i(t)}{c_i(t)} = \frac{1}{\sigma} (r(t) - (\delta + \rho))$$

Representative firm:

$$\max_{K, L} F(K, L) - rK - wL$$

FOCs: $r(t) = F_1(K, L) = f'(k)$ and

$w(t) = F_2(K, L) = f(k) - kf'(k)$.

Market Equilibrium = Planner's Solution

Equilibrium prices:

$$r(t) = f'(k(t)), \quad w(t) = f(k(t)) - k(t) f'(k(t))$$

Substituting into the household problem and using symmetry ($c_i = c$, $k_i = k$):

$$\begin{aligned} \dot{k}(t) &= f(k(t)) - \delta k(t) - c(t) \\ \frac{\dot{c}(t)}{c(t)} &= \frac{1}{\sigma} (f'(k(t)) - (\delta + \rho)) \end{aligned}$$

These are **identical** to the planner's equations.

⇒ **First and Second Welfare Theorems hold:** the competitive equilibrium is Pareto optimal.

Uncertainty: The Poisson Process

Motivation: Uncertainty in Continuous Time

Discrete time: natural to introduce “a shock each period.”

Continuous time: no natural unit of time. How to represent randomness?

Two broad approaches:

1. **Diffusion process** (Brownian motion): constant flow of small, frequent shocks
 - Central in finance and asset pricing
 - Requires stochastic calculus (beyond scope of the book)
2. **Jump process** (Poisson process): rare but discrete changes at random times
 - Mathematically accessible
 - Models: job loss, innovation, firm entry/exit

From Bernoulli Trials to the Poisson Distribution

Start in discrete time: Bernoulli trial with success probability $\lambda \in [0, 1]$ per year.

Divide the year into n subperiods, each with success probability λ/n :

- Expected successes in one year: still λ
- Distribution: Binomial with n trials, probability λ/n

$$b(k, n) = \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

Key cases:

$$b(0, n) = \left(1 - \frac{\lambda}{n}\right)^n$$

$$\frac{b(k, n)}{b(k-1, n)} = \frac{n - (k-1)}{k} \frac{\lambda/n}{1 - \lambda/n} = \frac{\lambda - (k-1)\lambda/n}{k - \lambda k/n}$$

Taking $n \rightarrow \infty$: The Poisson Distribution

As $n \rightarrow \infty$ (subperiod length $\rightarrow 0$):

$$p(0) \equiv \lim_{n \rightarrow \infty} b(0, n) = e^{-\lambda}$$

$$\lim_{n \rightarrow \infty} \frac{b(k, n)}{b(k-1, n)} = \frac{\lambda}{k}$$

Applying recursively for $k = 1, 2, 3, \dots$:

$$p(k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

This is the **Poisson distribution**: probability of k successes when the number of trials $\rightarrow \infty$ and probability per trial $\rightarrow 0$, while expected successes remain λ .

The Poisson Process

Definition: A stochastic process where the number of successes in any interval $(t, t + T]$ follows a Poisson distribution:

$$p(k) = e^{-\lambda T} \frac{(\lambda T)^k}{k!}$$

- Expected successes in $(t, t + T]$: λT
- As earlier, this is the limit of repeated independent trials in subintervals $\Delta = T/n$ with success prob $\lambda\Delta$, as $\Delta \rightarrow 0$

Key properties:

- **Memoryless:** outcomes in disjoint intervals are independent
- No need to assume $\lambda \leq 1$ (just start from small enough Δ)
- Probability of no success in $(t, t + T]$: $e^{-\lambda T}$
- When $\lambda(s)$ is time-varying: probability of no success is $e^{-\int_t^{t+T} \lambda(s) ds}$

Application: Employment and Unemployment Dynamics

A group of N workers. Let $e(t)$ = employed, $u(t)$ = unemployed, $e(t) + u(t) = N$.

- Job separation: Poisson process with rate $\sigma > 0$
 - Probability of keeping job for t periods: $e^{-\sigma t}$
 - Expected job duration: $1/\sigma$
- Job finding: unemployed worker finds job with probability λdt during interval dt

Law of motion (using LLN): $\dot{u}(t) = (N - u(t))\sigma - u(t)\lambda$

Steady state ($\dot{u} = 0$):

$$\bar{u} = \frac{\sigma N}{\sigma + \lambda}, \quad \text{unemployment rate: } \frac{\bar{u}}{N} = \frac{\sigma}{\sigma + \lambda}$$

Unemployment rate: increasing in σ , decreasing in λ . Convergence from any initial condition.

Summary

Key Takeaways

1. **Notation:** $X(t)$ instead of X_t ; $\dot{X}(t)$ for instantaneous rate of change; $e^{-\rho t}$ for discounting
2. **Log-derivative trick:** $\dot{X}/X = d \log X/dt$ — growth rates from logs
3. **Hamiltonian method:** Control $\Rightarrow \partial H/\partial c = 0$; State $\Rightarrow \partial H/\partial a + \dot{\mu} = 0$
4. **Continuous-time Euler:** $-\frac{u''c}{u'} \cdot \frac{\dot{c}}{c} = r - \rho$
5. **Solow model:** $\dot{\tilde{k}} = sf(\tilde{k}) - (\delta + \gamma + n)\tilde{k}$ — convergence
6. **Neoclassical model:** saddle path; modified golden rule $f'(k^{**}) = \delta + \rho$; market equilibrium = planner
7. **Poisson process:** limit of Bernoulli trials; memoryless; $p(k) = e^{-\lambda T} (\lambda T)^k / k!$