Chapter 13: Growth

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Outline

1. Motivation
2. Empirical Patterns
3. Growth and Development Accounting
4. Neoclassical Growth with Investment-Specific Technical Change
5. Confronting the Model with Data
6. Endogenous Growth
   - AK Model
   - Expanding Varieties (Romer, 1990)
   - Semi-Endogenous Growth (Jones, 1995)
   - Quality Ladders and Creative Destruction
7. Conclusion

Motivation

Why Study Growth?

- U.S. GDP per capita has grown at $\approx 1.5\%$ per year for **200 years**
- Process transformed living standards far beyond what economic indicators reveal
- Standards of living improved not only in advanced economies but also in developing ones—at differing rates
- Understanding growth is "of first-order importance to human welfare"

Lucas (1988): "Is there some action a government of India could take that would lead the Indian economy to grow like Indonesia's or Egypt's? If so, what, exactly? . . . The consequences for human welfare involved in questions like these are simply staggering: Once one starts to think about them, it is hard to think about anything else."

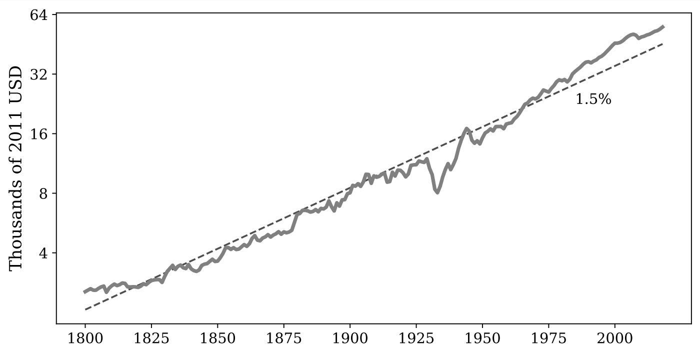
Chapter Roadmap

1. Empirical patterns: stylized facts about growth and development
2. Growth and development accounting
3. Neoclassical growth model with investment-specific technical change (ISTC)
4. Confronting the ISTC model with cross-country and time-series data
5. Endogenous growth theory
    - AK model
    - Expanding varieties (Romer, 1990)
    - Semi-endogenous growth (Jones, 1995)
    - Quality ladders and creative destruction (Aghion and Howitt, 1992)

Empirical Patterns

US Real GDP Per Capita

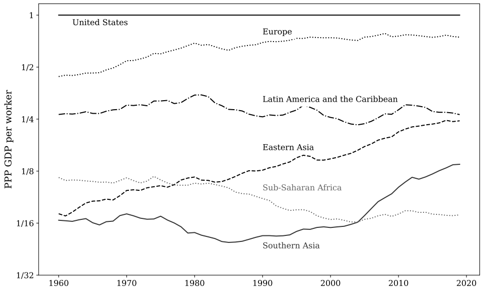
US Real GDP per Capita



- Straight line on log scale = constant growth rate
- Sustained growth began slowly ∼1,000 years ago, then accelerated after the Industrial Revolution
- Should not take such growth for granted

Cross-Country Growth: Regional Patterns

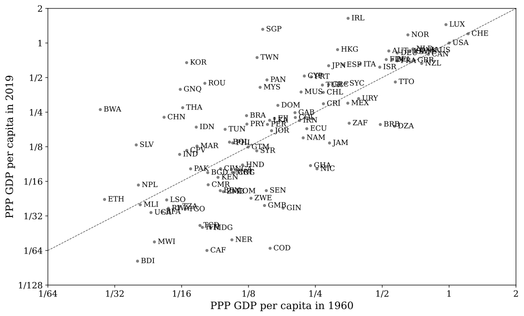
PPP GDP per Worker (US = 1), 1960-2019



- **Europe**: converged toward US (1960-1990), then parallel
- **Latin America**: mostly moved sideways
- **East Asia** (China, Japan, Korea): rose from $1/16$ to $\approx 1/4$
- **Sub-Saharan Africa**: grew with US until 1980, fell behind (from $1/8$ to $1/16$), then stabilized
- **South Asia** (led by India): tracked at $1/16$ until mid-1970s, surged to over $1/8$

Cross-Country Growth: Country-Level

PPP GDP per Capita in 1960 and 2019 (US = 1)



- Countries on the 45° line: same growth rate as U.S. — the "typical" pattern
- Growth miracles: South Korea, Singapore
- Growth disasters: Congo, Burundi
- Rich countries: clustered near U.S. growth rate
- Developing countries: much more dispersion

Convergence?

- The previous figure displays **neither strong divergence nor strong convergence**
- Two concepts:
  - **Beta convergence**: initially richer countries grow more slowly
  - **Sigma convergence**: narrowing of income dispersion over time
- The data show a lack of *both* unconditional beta and sigma convergence
- Absence of divergence is consistent with a common trend (e.g., technology that gradually diffuses across countries)
- Nath, Ramey and Klenow (2023): country *income differences* are persistent, whereas country *growth differences* are largely transitory

World Income Distribution

- Distribution shifts to the right over time (rapid growth in China and India)
- Left-skewed, but decreasingly so over time
- When each country is weighted equally, the distribution is less skewed

- Income differs by a factor of **64** between the Congo and the US
- China $\approx 1/4$; India $\approx 1/8$ of US per capita income

Growth and Development Accounting

Production Function Framework

Cobb-Douglas aggregate production function:

$$Y_{it} = K_{it}^{\alpha}(A_{it}H_{it})^{1-\alpha}$$

- Y: real output, K: physical capital, $H = hL$: total human capital (efficiency units summed across workers)
- A: residual labor-augmenting TFP, α: capital elasticity
- Human capital tied to Mincerian wage returns to schooling (Klenow and Rodriguez-Clare, 1997; Hall and Jones, 1999; Bils and Klenow, 2000)

Dividing by L and rearranging:

$$\frac{Y_{it}}{L_{it}} = \left(\frac{K_{it}}{Y_{it}}\right)^{\frac{\alpha}{1-\alpha}} \frac{H_{it}}{L_{it}} A_{it}$$

Why Use K/Y Instead of K/L?

- BGP of neoclassical model: **stable** K/Y that does not depend on levels of A or H/L
- Mankiw, Romer and Weil (1992) model: stationary level of H/L that does not depend on A or K/Y
- In the data: substantial persistence in K/Y within countries
- Countries do *not* converge to a common K/Y
- Using K/Y decomposition attributes a **larger role to TFP and HC** relative to physical capital
- α approximated by physical capital's **cost share** (not income share, to avoid markup bias)

Growth Accounting: Methodology

Taking logs and differentiating with respect to time:

$$\Delta \log\left(\frac{Y_i}{L_i}\right) = \frac{\alpha}{1-\alpha}\Delta \log\left(\frac{K_i}{Y_i}\right) + \Delta \log\left(\frac{H_i}{L_i}\right) + \Delta \log(A_i)$$

- Average the RHS components over time
- Back out TFP growth $\Delta \log(A)$ as a residual
- Compare contributions to LHS $\Delta \log(Y/L)$

Growth Accounting for the US

Growth accounting (annual %). $\alpha = 0.34$

	Contributions from			
Period	Y/L	K/Y	H/L	A
1948–2020	2.37	0.21	0.38	**1.79**
1948–1973	3.28	−0.18	0.27	**3.19**
1973–1995	1.54	0.46	0.36	**0.72**
1995–2007	2.80	0.32	0.40	**2.08**
2007–2020	1.64	0.43	0.59	**0.63**

- K/Y: modest contribution ($< 1/10$ of average growth)
- H/L: twice as much (primarily rising years of education)
- TFP A: $\approx \mathbf{75}\%$ of growth — echoes Solow (1957)

TFP Drives Medium-Run Growth Shifts

- 1948–1973: high TFP growth (3.19%)
- 1973–1995: "productivity slowdown" (0.72%)
- 1995–2007: TFP revival (2.08%)
  - Jorgenson, Ho, and Stiroh (2008): ICT contribution, both directly (ICT-producing sector) and downstream (ICT-using sectors like retail)
  - Resolved the **Solow Paradox**: "you can see the computer age everywhere but in the productivity statistics" (Solow, 1987)
- 2007–2020: another slowdown (0.63%)

Takeaway: When growth is high or low, it is primarily due to the pace of TFP growth.

Development Accounting Across Countries (2019)

Development accounting, 117 countries

	Contributions from			
Statistic	Y/L	K/Y	H/L	A
Variance of log	1.00	0.14	0.08	0.57
Elasticity wrt Y/L	—	0.14	0.22	**0.64**
90/10 ratio	12.00	1.40	1.74	4.92

- Development accounting: the same equation, but level instead of growth rate
- Elasticities add up to 1 by construction (covariance split equally)
- Capital intensity: **14%**; human capital: **22%**; residual TFP: **64%**
- 90/10 ratio check: $\log(4.92)/\log(12) \approx 0.64$ — consistent

Improving Human Capital Measurement

- Schoellman (2012): incorporating differences in the **quality** of schooling
- Lagakos, Moll, Porzio, Qian & Schoellman (2018): human capital accumulated **on the job**
- Result: HC contribution rises to $\approx 50\%$, TFP contribution winnows to $\approx 40\%$

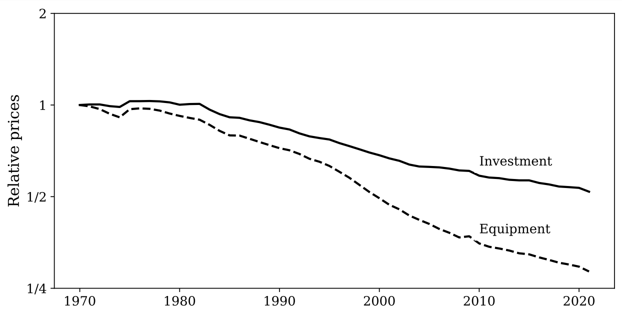
Elasticities broadly similar in 1960 and 2019, though:

- HC importance has diminished over time
- Role of capital intensity fell from 1990 to 2010 before rising

Neoclassical Growth with Investment-Specific Technical Change

Motivation: Declining Relative Price of Investment

Relative price of equipment and investment in the US

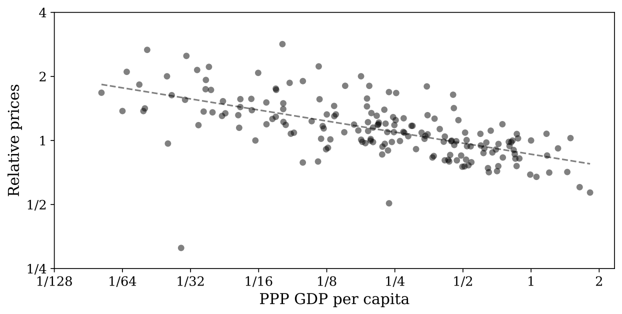


- Relative price of **investment**: fell at $\approx 1.3\%$/year since 1970
- **Equipment**: fell at $\approx 2.5\%$/year
- **Structures**: nearly doubled ($+1.2\%$/year relative to consumption)

$\Rightarrow$ Motivates **investment-specific technical change** (ISTC): Greenwood, Hercowitz, and Krusell (1997).

Cross-Country Relative Prices of Investment

Price of investment relative to consumption across countries, 2019



- Poorest countries: investment is $\approx 2\times$ more expensive
- Poorest countries particularly inefficient at producing investment goods (Hsieh and Klenow, 2007)
- Consequence: richer countries with the same nominal saving rate arrive at a higher real K/Y
- This has implications for development accounting

Model: Preferences

Representative household, continuous time, abstracting from endogenous labor supply:

$$\max_{\{a(t),c(t)\}} \int_0^\infty e^{-(\rho-n)t} \frac{c(t)^{1-\sigma}-1}{1-\sigma} dt$$

subject to

$$\dot{a}(t) = (r(t)-n)a(t) + w(t)h - c(t), \quad a(0) \text{ given}$$

and the no-Ponzi game condition:

$$\lim_{T\to\infty} e^{-\int_0^T (r(\tau)-n)d\tau} a(T) \geq 0.$$

- ρ: discount rate, n: population growth, σ: CRRA parameter
- h: human capital per worker (constant), c: per-capita consumption
- Price of consumption good normalized to 1

Model: Return Condition

Household converts savings into investment good at price P_x (one unit of consumption good a becomes $1/P_x$ units of investment good K) and rents capital to firms.

Return condition, which can be viewed as the asset-pricing equation for one unit of capital (whose value is $P_x(t)$), links the return per one unit of $a(t)$ to the rental rate $R(t)$, depreciation, and capital gains:

$$r(t)P_x(t) = R(t) - \delta P_x(t) + \dot{P}_x(t).$$

Note: This equation can also be derived from the consumer's capital accumulation equation $\dot{K}(t) = I(t)/P_x(t) - \delta K(t)$, where the investment $I(t) = R(t)K(t) + w(t)he^{nt} - c(t)e^{nt}$. This capital accumulation equation can be rewritten as, using $K(t) = a(t)e^{nt}/P_x(t)$,

$$\dot{a}(t) = \left(\frac{R(t)}{P_x(t)} - \delta + \frac{\dot{P}_x(t)}{P_x(t)} - n \right) a(t) + w(t)h - c(t).$$

The first term inside the parenthesis corresponds to $r(t)$.

Model: Technology

Final output (competitive, Cobb-Douglas):

$$Y(t) = K(t)^{\alpha}\big(A_y e^{\gamma_y t} L(t)\big)^{1-\alpha}$$

- $\alpha \in (0, 1)$: output elasticity of capital
- $A_y e^{\gamma_y t}$: Harrod-neutral (labor-augmenting) technical change
- $\gamma_x \geq 0$: **investment-specific** technical change

Output Y can be transformed:

- 1-for-1 into consumption goods
- 1-for-$A_x e^{\gamma_x t}$ into investment goods

Both technologies linear $\Rightarrow$ $P_c(t) = 1$ and $P_x(t) = e^{-\gamma_x t}/A_x$.

Y is measured in consumption units (GDP deflated by consumption price).

Model: Firm Problem and Equilibrium Definition

Firm solves a static profit maximization:

$$\pi(t) = \max_{K,L} \left\{ K^\alpha \left(A_y e^{\gamma_y t} L\right)^{1-\alpha} - R(t)K - w(t)L \right\}$$

A competitive equilibrium is a path of prices and quantities satisfying:

1. Solves household problem and firm problem
2. Market clearing: $K(t)\, e^{-\gamma_x t}/A_x = a(t)\, e^{nt}$ and $L(t) = e^{nt}h$
3. Return condition: $r(t) = R(t)\, A_x\, e^{\gamma_x t} - \delta - \gamma_x$

Boundedness condition (utility is finite):

$$n - \rho + \frac{\alpha(1-\sigma)}{1-\alpha}\,\gamma_x + (1-\sigma)\gamma_y < 0$$

This is because the per capita consumption grows at the rate $\alpha\gamma_x/(1-\alpha) + \gamma_y$, as we will see later.

Equilibrium Dynamics

Equilibrium reduces to two ODEs in $K(t)$ and $c(t)$:

Consumption Euler equation:

$$\frac{\dot{c}(t)}{c(t)} = \frac{\alpha A_x e^{\gamma_x t}\left(\frac{A_y\, e^{(\gamma_y+n)t} h}{K(t)}\right)^{1-\alpha} - \delta - \gamma_x - \rho}{\sigma}$$

Capital accumulation:

$$\frac{\dot{K}(t)}{K(t)} = A_x\left(\left(\frac{e^{(\gamma_x/(1-\alpha)+\gamma_y+n)t} A_y h}{K(t)}\right)^{1-\alpha} - \frac{e^{(\gamma_x+n)t} c(t)}{K(t)}\right) - \delta$$

Initial condition: $K(0)$ given. Terminal condition: transversality condition

$$\lim_{T\to\infty}\left\{\frac{K(T)}{A_x} e^{-(\gamma_x+\rho)T} c(T)^{-\sigma}\right\} = 0$$

Balanced Growth Path: Growth Rates

A balanced growth path: all prices and quantities grow at constant rates.

Can the RHS of $\dot{K}/K$ equation be constant? The standard candidate $\dot{K}/K = \gamma_y + n$ does *not* work. Instead:

$$g_K \equiv \frac{\dot{K}}{K} = \frac{\gamma_x}{1-\alpha} + \gamma_y + n$$

$$g_c \equiv \frac{\dot{c}}{c} = \frac{\alpha\gamma_x}{1-\alpha} + \gamma_y$$

- Capital grows faster than in the standard case because investment gets cheaper
- Cobb-Douglas is **essential** for a BGP with ISTC (relates to Uzawa's theorem)

Balanced Growth Path: Detrended System

Define $\tilde{c}(t) \equiv c(t)/e^{g_c t}$ and $\tilde{k}(t) \equiv K(t)/e^{g_K t}$. The detrended system:

$$\frac{\dot{\tilde{c}}}{\tilde{c}} = \frac{\alpha A_x \left(\frac{A_y h}{\tilde{k}}\right)^{1-\alpha} - \delta - \gamma_x - \rho}{\sigma} - \frac{\alpha \gamma_x}{1-\alpha} - \gamma_y$$

$$\frac{\dot{\tilde{k}}}{\tilde{k}} = A_x \left(\left(\frac{A_y h}{\tilde{k}}\right)^{1-\alpha} - \frac{\tilde{c}(t)}{\tilde{k}(t)} \right) - \delta - \frac{\gamma_x}{1-\alpha} - \gamma_y - n - \frac{\tilde{c}}{\tilde{k}}$$

Admits a stationary point $(\tilde{c}^*, \tilde{k}^*)$. The terminal condition is fulfilled under our assumption.

Balanced Growth Path: Steady-State Values

Setting $\dot{\tilde{k}} = \dot{\tilde{c}} = 0$:

$$\tilde{k}^* = A_x^{\frac{1}{1-\alpha}} A_y h \left(\frac{\alpha}{\frac{\alpha\sigma\gamma_x}{1-\alpha} + \gamma_x + \sigma\gamma_y + \delta + \rho} \right)^{\frac{1}{1-\alpha}}$$

$$\tilde{c}^* = (\tilde{k}^*)^\alpha (A_y h)^{1-\alpha} - (g_K + \delta) \frac{\tilde{k}^*}{A_x}$$

Endogenous saving rate: $s^* = \alpha(g_K + \delta)/(\sigma g_c + \gamma_x + \delta + \rho)$, constant on BGP.

Golden Rule: $\tilde{k}^{gold}$ maximizes $\tilde{c}^*$, achieved when $s^{gold} = \alpha$. Since our assumption ensures $\sigma g_c + \gamma_x + \delta + \rho > g_K + \delta$:

$$\tilde{k}^* < \tilde{k}^{gold} \quad \text{(no dynamic inefficiency)}$$

BGP: Predictions for Prices and Quantities

Along the balanced growth path:

- **Wage** w: grows at rate g_c
- **Interest rate**: $r^* = \sigma g_c + \rho$ — **constant**
- **Rental rate** R: falls at rate γ_x (investing one unit of capital gets cheaper in consumption units)
- **Nominal capital-output ratio** $P_x K/Y = s^*/(g_K + \delta)$: constant
- **Real capital-output ratio** K/Y: grows at rate $(1 - s^*)\gamma_x$

GDP deflator growth rate (defined as $(\dot{P}_x/P_x)^{s^*}(\dot{P}_c/P_c)^{1-s^*}$): $-(1 - s^*)\gamma_x$ (investment share pushes deflator down).

Transitional Dynamics

- Similar to the standard neoclassical model
- If $\tilde{k}(0) < \tilde{k}^*$: capital accumulates faster, $\tilde{k}$ increases until reaching $\tilde{k}^*$
- **Special case** $\sigma = \alpha$: closed-form consumption rule exists
- General case can be solved by linearizing around the steady state
- Simple calibration: **half-life** ≈ 5.5 **years** (fast convergence)

Discussion of the Framework

- **Closed economy**: justified for the world; trade small for large economies
- **Representative household**: CRRA ⇒ homothetic utility ⇒ aggregation holds with heterogeneity
- **CRRA**: key for constant long-run saving rate (with growth)
- **No endogenous labor supply**: can be added, reconciled with BGP (Boppart and Krusell, 2020)
- **Constant** h: model can study transitional changes; sustained h growth would change dynamics fundamentally
- **Structural change**: along BGP, nominal investment share is constant ⇒ no sectoral reallocation. Can be reconciled with aggregate BGP (Kongsamut et al., 2001; Ngai and Pissarides, 2007; Boppart, 2014)
- **Energy, natural resources**: see Chapter 25

Confronting the Model with Data

Income Decomposition in Consumption Units

Output per worker (in consumption units) along the BGP:

$$\frac{Y(t)}{e^{nt}} = \left(\frac{P_k(t)K(t)}{Y(t)}\right)^{\frac{\alpha}{1-\alpha}} \cdot h \cdot \underbrace{A_y A_x^{\frac{\alpha}{1-\alpha}}}_{=A} \cdot e^{(\gamma_y + \gamma_x \frac{\alpha}{1-\alpha})t}$$

where $P_k(t) = A_x^{-1} e^{-\gamma_x t}$.

The nominal K/Y ratio is constant on the BGP and **independent** of levels of h, A_y, or A_x.

Key: This decomposition is theory-consistent—uses nominal K/Y rather than a "real" measure.

Two-Sector Development Accounting

Results:

- Residual TFP A: **72%** of income differences (vs. 64% in the earlier dev accounting)
- Physical capital: only **7%** (vs. 14%)
- Human capital: **22%** (unchanged)

The message that most income differences are unaccounted for by physical and human capital is **reinforced** because the relative price of capital is higher in poorer countries.

Decomposing A: Using relative prices P_x/P_c series:

- Elasticity of $A_x^{\alpha/(1-\alpha)}$ with respect to $(P_Y Y)/(P_C L)$: 0.10
- $\Rightarrow$ About **14% of the A differences** are due to investment-specific TFP A_x

Can Transitional Dynamics Explain Growth Miracles?

Japan: GDP per capita rose from ∼20% to ∼80% of U.S. (1950–1990).

If purely capital accumulation:

- Need $(k^{JP}/k^{US})^{\alpha} = 0.2$ in 1950
- Implied rental rate ratio: $R^{JP}/R^{US} = 0.2^{-(1-\alpha)/\alpha}$
- With $\alpha = 1/3$: Japan's rental rate would be **25 times** the U.S. rate in 1950!
- By 1990: ratio should have declined to < 2
- Such a dramatic decline was **not observed**

⇒ Capital deepening alone **cannot** explain growth miracles.

⇒ More realistic: episodes of fast catch-up growth reflect transitional changes in A_y or A_x.

U.S. Growth Decomposition in Consumption Units

- U.S. **nominal** K/Y: remarkably stable over the post-war period
- Modest capital deepening in **real** K/Y (especially since 1970s)
- Human capital contribution largely unchanged
- Clear majority from TFP residual A

Decomposing U.S. TFP growth:

- Per-capita consumption growth: $2.07\%/\text{year} = \gamma_y + \gamma_x \frac{\alpha}{1-\alpha}$
- $\gamma_x = 0.013$
- With $\alpha \approx 1/3$: $\gamma_y = 0.014$
- $\Rightarrow \approx$ **31% of per-capita consumption growth** from ISTC

Endogenous Growth

The AK Model: Setup

Simplest endogenous growth theory. Set $\alpha = 1$, $A_x = 1$, $\gamma_y = \gamma_x = 0$:

$$Y(t) = AK(t)$$

Planner's problem:

$$\max \int_0^\infty e^{-(\rho-n)t} \frac{c(t)^{1-\sigma} - 1}{1-\sigma} dt$$

subject to $\dot{K} \leq AK - e^{nt}c - \delta K$ and $K(t) \geq 0$.

Requires $\sigma(A - \delta - n) > A - \delta - \rho$ for bounded utility.

The AK Model: Solution and Properties

Euler equation:

$$\frac{\dot{c}}{c} = \frac{A - \delta - \rho}{\sigma}$$

- Growth rate is **constant** irrespective of initial K — **no transition dynamics**
- Consumption rule:

$$c(t) = (A - \delta - n - (A - \delta - \rho)/\sigma)e^{-nt}K(t)$$

- Capital grows at rate $n + (A - \delta - \rho)/\sigma$
- Transversality condition selects the unique value of $c(0)$
- Can be decentralized with competitive markets

Endogenous growth: rate depends on preferences (ρ, σ), technology (A), **and policy** (e.g., capital income tax τ monotonically reduces growth).

The AK Model: Critique

- $\alpha = 1$: no role for labor (labor earns ∼2/3 of all income!)
- Would need massive IRS (Romer, 1986) to reconcile with empirical $\alpha \approx 1/3$
- **AK as reduced form**: $Y = F(K, H)$ with CRS in K and H combined
  - But does H grow at a steady rate? Requires increasing investment per person (rising schooling)
  - Mincerian returns to education/experience: no secular trend
  - Bils and Klenow (2000): schooling has a **level effect**, not a **growth effect**
  - Klenow and Rodriguez-Clare (2005): schooling levels correlate with income *levels*, not growth *rates*
- Entirely factor-accumulation based ⇒ zero TFP growth if inputs properly measured

Romer (1990): Aggregate Production Function

$$Y(t) = \frac{A}{1-\phi} L_y(t)^{\phi} \int_0^{N(t)} x(\nu, t)^{1-\phi} \, d\nu$$

- L_y: labor in final good production
- $x(\nu)$: quantity of machine variety ν
- N: number of varieties (knowledge stock)
- $\phi \in (0,1)$
- CRS in $(L_y, \{x(\nu)\})$, but **IRS including** N (love-of-variety)
- Can be viewed as Cobb-Douglas over L_y and a CES machine composite with $\epsilon = 1/\phi$
- Machines depreciate fully after use (like materials)

Romer (1990): Final Good Producer

Final good is competitively produced. The representative firm solves:

$$\max_{L_y, \{x(\nu)\}} \left\{ \frac{A}{1-\phi} \, L_y^{\phi} \int_0^N x(\nu)^{1-\phi} \, d\nu - wL_y - \int_0^N p(\nu)x(\nu) \, d\nu \right\}$$

First-order conditions:

$$w = \phi \, Y / L_y$$

$$p(\nu) = A \, L_y^{\phi} \, x(\nu)^{-\phi}, \quad \forall \nu$$

Romer (1990): Machine Producers

Each variety ν produced by one monopolist. Marginal cost: ψ units of final output.

$$\max_{p(\nu),x(\nu)} \; p(\nu)\, x(\nu) - \psi\, x(\nu) \quad \text{s.t. the demand } p(\nu) = A L_y^{\phi} x(\nu)^{-\phi}$$

Optimal price (constant markup over marginal cost):

$$p(\nu) = \frac{\psi}{1-\phi}, \quad \forall \nu$$

Optimal quantity:

$$x(\nu) = \left(\frac{A(1-\phi)}{\psi}\right)^{1/\phi} L_y, \quad \forall \nu$$

Profit:

$$\pi(\nu) = \phi A \left(\frac{A(1-\phi)}{\psi}\right)^{1/\phi - 1} L_y$$

Increasing in market size L_y.

Romer (1990): Innovation and Entry

Ideas production function (hiring $1/(\eta N)$ labor invents one variety):

$$\dot{N}(t) = \eta\, N(t)\, L_r(t), \quad N(0) > 0$$

N term: **proportional knowledge spillover**—more varieties make R&D more productive.

Inventor receives a **perpetual patent**. Patent value:

$$V(t) = \int_t^\infty e^{-\int_t^s r(\tau)\, d\tau}\, \pi(s)\, ds$$

HJB: $V(t)\, r(t) = \pi(t) + \dot{V}(t)$.

Free entry in R&D

$$V(t) = \frac{w(t)}{\eta\, N(t)}$$

Romer (1990): Household

No population growth. Representative household:

$$\max \int_0^\infty e^{-\rho t} \frac{c(t)^{1-\sigma} - 1}{1-\sigma} \, dt$$

subject to $\dot{a} = ra + wL - c$ and no-Ponzi game condition.

Wealth: $a(t) = \int_0^{N(t)} V(\nu, t) \, d\nu$ (owns all firms).

Euler equation:

$$\frac{\dot{c}}{c} = \frac{r - \rho}{\sigma}$$

Boundedness of utility:

$$\rho > (1-\sigma) \frac{(1-\phi)\eta L - \rho}{1 - \phi + \sigma}$$

Positive growth:

$$\rho < (1-\phi)\eta L$$

Romer (1990): Aggregation

Using symmetry (x, p same for all ν):

Output:

$$Y(t) = \frac{A}{1-\phi}\left(\frac{A(1-\phi)}{\psi}\right)^{\frac{1-\phi}{\phi}} L_y(t)N(t)$$

Wage:

$$w(t) = \frac{\phi A}{1-\phi}\left(\frac{A(1-\phi)}{\psi}\right)^{\frac{1-\phi}{\phi}} N(t)$$

Value of a product:

$$V(t) = \frac{\phi A}{\eta(1-\phi)}\left(\frac{A(1-\phi)}{\psi}\right)^{\frac{1-\phi}{\phi}}$$

Interest rate from HJB:

$$r(t) = \frac{\pi(t)}{V(t)} = \eta(1-\phi)\, L_y(t)$$

Romer (1990): Balanced Growth Path

System reduces to a single ODE in L_y:

$$\frac{\dot{L}_y}{L_y} = \frac{\eta(1-\phi+\sigma)L_y - \rho}{\sigma} - \eta L$$

No transition dynamics (from the boundary conditions)

Unique constant solution:

$$L_y^* = \frac{\sigma L + \rho/\eta}{1-\phi+\sigma}, \qquad L_r^* = \frac{(1-\phi)L - \rho/\eta}{1-\phi+\sigma}$$

Endogenous growth rate:

$$g^* = \frac{\dot{N}}{N} = \frac{\dot{Y}}{Y} = \frac{\dot{c}}{c} = \frac{\dot{a}}{a} = \frac{(1-\phi)\eta L - \rho}{1-\phi+\sigma}$$

Interest rate:

$$r^* = (1-\phi)\frac{\eta\sigma L + \rho}{1-\phi+\sigma}$$

Romer (1990): Scale Effect and Policy Implications

Comparative statics of g^*:

- Increasing in $L \Rightarrow$ **strong scale effect**: bigger countries grow faster, higher R&D intensity
- Increasing in η (R&D efficiency)
- Decreasing in ρ (impatience) and σ (curvature)

Problem: Scale effect is **counterfactual**:

- Cross-section: bigger countries do not grow faster
- Time series: R&D investment trends up, but growth rate does not

Policy: Taxing profits, lowering markups via antitrust, or expiring patents all monotonically **lower** growth. Policy implications are stark.

Romer (1990): Social Planner

Planner maximizes utility subject to resource constraint and $\dot{N} = \eta N(L - L_y)$.

Planner's growth rate:

$$g^{SP} = \frac{\eta L - \rho}{\sigma}$$

Two sources of inefficiency:

1. **Monopoly distortion**: machines underused by factor $(1-\phi)^{1/\phi} < 1$
2. **Knowledge externality**: spillovers in $\dot{N} = \eta N L_r$ not internalized by innovators (R&D effort determined only by discounted profits)

In general: $g^{DE} < g^{SP}$ (too little R&D).

Semi-Endogenous Growth: Setup

Two modifications to the Romer model (Jones, 1995):

1. Less-than-proportional knowledge spillovers:

$$\dot{N}(t) = \eta\, N(t)^{\epsilon}\, L_r(t), \quad N(0) > 0$$

- $0 < \epsilon < 1$: positive but limited spillovers
- $\epsilon < 0$: "fishing out" effect
- $\epsilon = 0$: no spillovers; $\epsilon = 1$: back to Romer

2. Population growth: $L(t) = Le^{nt}$, $n > 0$

Boundedness of utility: $\rho - n > n/(1-\epsilon)$.

Semi-Endogenous Growth: Balanced Growth Path

Along BGP, L_y and L_r both grow at rate n (otherwise shares can't be constant).

BGP requires

$$\frac{\dot{N}}{N} = \frac{n}{1-\epsilon}$$

- Growth determined by n and ϵ **only**
- Without population growth, **no sustained long-run growth**
- DRS in ideas production: only way to sustain growth is growing number of researchers
- Long-run growth rate is **efficient** (planner gets same rate)
- No role for policy on the long-run growth rate
- Consistent with stable U.S. growth rate + growing researcher population

Semi-Endogenous Growth: Transition and Level Effects

Transition dynamics:

$$\frac{n}{\eta(1-\epsilon)} = N(t)^{\epsilon-1}(Le^{nt} - L_y(t))$$

Together with free entry, pins down $N(t)$ supporting BGP. If initial N deviates: transitional dynamics. Can be **slow** (if ϵ close to 1; Bloom, Jones and Van Reenen, 2020: $\epsilon \approx 0.8$ in semiconductors).

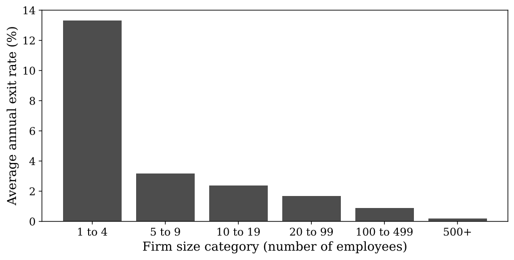
Level effect of policy: Along BGP:

$$\frac{Y(t)}{L_y(t)} = \bar{A}\left(\frac{L_r(t)}{L(t)}\right)^{\frac{1}{1-\epsilon}} L(t)^{\frac{1}{1-\epsilon}}$$

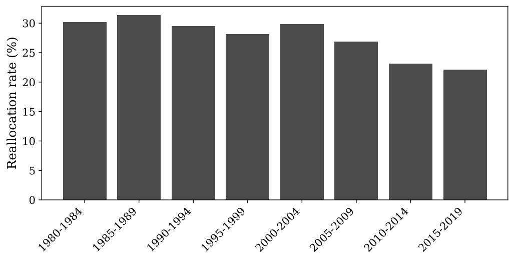
- No scale effect on growth, but scale effect on **levels**
- Policy can affect R&D intensity $\Rightarrow$ affects long-run income level

Motivation: Firm Exit and Job Reallocation

Firm exit rates in the U.S., 1980–2022, by size category

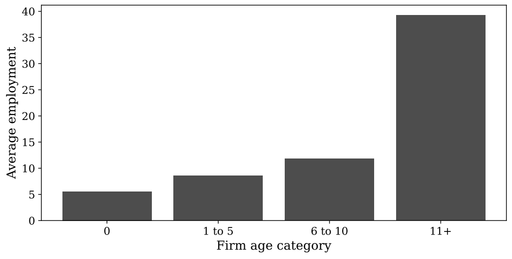


Job reallocation in the U.S., 1980–2019 (5-year periods)

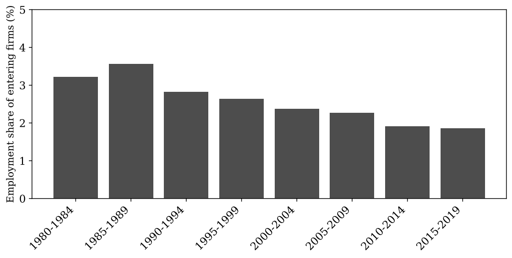


More Facts on Firm Dynamics

Average employment by firm age category, 1988–2022

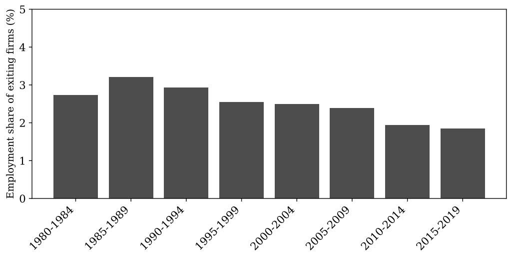


Entrant employment share, 1980–2019



More Facts on Firm Dynamics

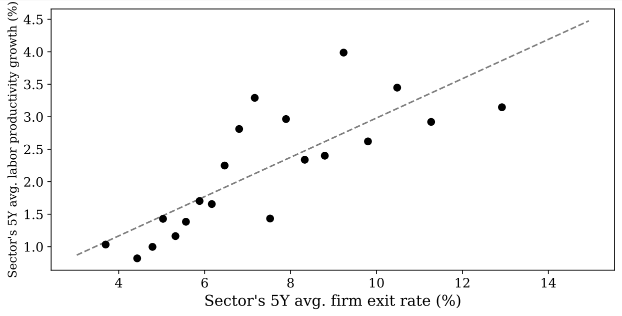
Exiting firm employment share, 1980–2019



- "Declining business dynamism": Decker, Haltiwanger, Jarmin and Miranda (2016); Akcigit and Ates (2023)

Growth and Exit Rates

Sector productivity growth vs. firm exit rate, 1988–2022



- Consistent with creative destruction: industries with **high exit rates** tend to exhibit **faster productivity growth**

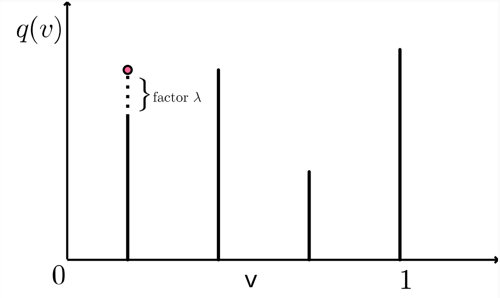
Quality Ladders: Setup

Final good (competitive, fixed measure 1 of varieties):

$$Y(t) = \frac{A}{1-\phi} L(t)^{\phi} \int_0^1 q(\nu,t)\, x(\nu,t)^{1-\phi}\, d\nu$$

Quality evolves in discrete steps:

$$q(\nu,t) = \lambda^{m(\nu,t)}\, q(\nu,0), \qquad \lambda > 1$$



Quality Ladders: Setup

Resource constraint:

$$Y = C + X + Z$$

- intermediate goods

$$X = \int_0^1 \psi \, q(\nu, t) \, x(\nu, t) \, d\nu$$

- aggregate research effort ("lab equipment" model)

$$Z = \int_0^1 Z(\nu, t) \, d\nu$$

Quality Ladders: Innovation Technology

Innovation rate on product line ν:

$$z(\nu,t) = \eta \, \frac{Z(\nu,t)}{q(\nu,t)}$$

- $\eta > 0$: R&D efficiency
- Cost of innovating **scales with quality**: harder to improve better products
- Each innovation: quality jumps by factor $\lambda > 1$
- New innovator **takes over** the product line (creative destruction)

Preferences:

$$U(0) = \int_0^\infty e^{-\rho t} \frac{C(t)^{1-\sigma} - 1}{1-\sigma} \, dt$$

Fixed labor supply

$$L(t) = L.$$

Quality Ladders: Final Good Producer

Final good producer maximizes:

$$\Pi = \frac{A}{1-\phi} L^{\phi} \int_0^1 q(\nu)\, x(\nu)^{1-\phi}\, d\nu - \int_0^1 p(\nu)\, x(\nu)\, d\nu - wL$$

First-order conditions:

$$w = \phi Y/L$$

$$x(\nu, t) = \left(\frac{A\, q(\nu, t)}{p(\nu, t)}\right)^{1/\phi} L$$

Intermediate demand: inversely related to quality-adjusted price, increasing in L.

Quality Ladders: Intermediate Good Monopolists

Assume **drastic innovations**: $\lambda \geq \left(\frac{1}{1-\phi}\right)^{(1-\phi)/\phi}$
$\Rightarrow$ Monopolist not constrained by lower-quality competitors. (as opposed to limit pricing with incremental innovations.)

Profit-maximizing price:

$$p(\nu, t) = \frac{\psi}{1-\phi}\, q(\nu, t)$$

Input demand (independent of ν and t!):

$$x(\nu, t) = \left(\frac{A(1-\phi)}{\psi}\right)^{1/\phi} L, \quad \forall \nu, t$$

Profit (proportional to quality):

$$\pi(\nu, t) = \phi A \left(\frac{A(1-\phi)}{\psi}\right)^{\frac{1-\phi}{\phi}} q(\nu, t)\, L$$

Quality Ladders: Research Firms

Free entry into research. Research firm chooses effort for each variety:

$$\max_{Z(\nu,t)} \frac{\eta\, Z(\nu,t)}{q(\nu,t)}\,\lambda\, V(\nu,t) - Z(\nu,t)$$

Innovator takes over entire profit stream (creative destruction) and earns profits scaled up by λ.

Free entry condition:

$$\frac{\eta}{q(\nu,t)}\,\lambda\, V(\nu,t) = 1$$

Quality Ladders: Arrow Replacement Effect

Would incumbents invest to improve their own products?

Incumbent gains only the *increment* to value:

$$\frac{\eta}{q(\nu,t)}\,(\lambda-1)\,V(\nu,t) < \frac{\eta}{q(\nu,t)}\,\lambda\,V(\nu,t) = 1$$

$\Rightarrow$ Incumbents have **less incentive** to innovate than entrants ($\lambda - 1 < \lambda$).

This is the **Arrow Replacement Effect**.

To explain the reality that incumbents *do* improve their own products: could allow lower incumbent innovation costs. With convex costs: both incumbent and entrant innovation in equilibrium.

Quality Ladders: Patent Value

$$V(\nu,t) = \int_t^\infty e^{-\int_t^s r(s')\,ds'}\, e^{-\int_t^s z(\nu,s')\,ds'}\, \pi(\nu,s)\,ds$$

Discounted by **both** the interest rate **and** the probability of creative destruction.

Free entry $\Rightarrow V(\nu,t)/q(\nu,t) = 1/(\lambda\eta)$ for all ν, t.

No transition dynamics (r constant) and π proportional to $q \Rightarrow$ Constant V/q means $z(\nu,t) = z^*$ for all ν, t (common, constant rate).

Household: standard Euler equation $\dot{c}/c = (r-\rho)/\sigma$ and transversality condition. Assets $= \int_0^1 V(\nu,t)\,d\nu$.

Quality Ladders: Balanced Growth Path

Average quality $Q(t) = \int_0^1 q(\nu, t)\, d\nu$ grows at a smooth rate despite stochastic individual $q(\nu, t)$:

$$\frac{\dot{Q}}{Q} = (\lambda - 1)\, z^* \equiv g^*$$

(Uses a version of the Law of Large Numbers: integral of continuum of i.i.d. random variables equals the expectation.)

Three BGP equations:

$$r^* = \sigma g^* + \rho$$

$$r^* + z^* = \lambda \eta \phi A \left(\frac{A(1-\phi)}{\psi} \right)^{\frac{1-\phi}{\phi}} L$$

$$g^* = (\lambda - 1)\, z^*$$

Quality Ladders: Growth Rate

Solving the three equations:

$$g^* = \frac{\lambda\eta\phi A\left(\frac{A(1-\phi)}{\psi}\right)^{\frac{1-\phi}{\phi}} L - \rho}{\sigma + 1/(\lambda - 1)}$$

Comparative statics: The growth rate is

- increasing in λ (step size), η (R&D efficiency), L (**strong scale effect**)
- decreasing in ρ, σ, ψ

Same strong scale effect as Romer; could make semi-endogenous with DRS in innovation

Quality Ladders: Social Planner

Planner growth rate:

$$g^{SP} = \frac{(\lambda - 1)\eta\phi A(1-\phi)^{-1}(A/\psi)^{\frac{1-\phi}{\phi}} L - \rho}{\sigma}$$

1. **Business stealing**: Private innovators reap full λ; social gain only $\lambda - 1$
   - Force for $g^{SP} < g^{DE}$ (too much private R&D)
2. **Markup distortion**: Planner uses intermediates more intensively (no markup)
   - Force for $g^{SP} > g^{DE}$ (too little private R&D)
3. **Knowledge externality**: Planner sees innovation as lasting forever; private profits truncated by creative destruction
   - Force for $g^{SP} > g^{DE}$ (too little private R&D)

Net: Ambiguous, but calibrations typically find $g^{SP} \gg g^{DE}$

Conclusion

Key Takeaways

1. **Empirics**: U.S. grew at $\sim$1.5%/year for 200 years; income differs 64$\times$ across countries; no unconditional convergence
2. **Growth accounting**: TFP $\approx$ 75% of U.S. growth; $\approx$ 64% of cross-country income differences
3. **Neoclassical + ISTC**: Relative price of investment falling; $\sim$31% of per-capita growth from ISTC; capital deepening alone cannot explain growth miracles
4. **AK model**: Endogenous growth but knife-edge
5. **Romer**: R&D-driven expanding varieties; $g^{DE} < g^{SP}$; counterfactual scale effect
6. **Semi-endogenous**: $g = n/(1-\epsilon)$; no scale effect on growth; slow transitions
7. **Quality ladders**: Creative destruction; business stealing vs. knowledge externalities; more nuanced policy trade-offs than Romer

Open Questions and Frontier

- Connecting aggregate growth to **firm dynamics** (entry, exit, life-cycle growth) and **market structure** (concentration, market power)
- **Markups**: source of misallocation *and* incentive for innovation (Chapter 22)
- Mapping endogenous growth theories to microeconomic data on firms and products
- **Structural change**: reconciling unbalanced sectoral growth with balanced aggregates
- International technology diffusion and adoption
- Role of energy, natural resources, and climate (Chapter 25)