Chapter 20: Frictional Labor Markets

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Outline

Introduction

Motivation

Early real business cycle models assumed a **frictionless labor market**:

- All firms find workers; all workers find jobs
- Equilibrium wage equates labor demand and supply
- No role for unemployment

But unemployment is a central feature of the business cycle:

- Unemployment rate rises sharply in recessions
- Elevated unemployment is among the most important social costs of recessions
- Policies (UI, job training) aim to reduce unemployment
- Need a formal framework where unemployment arises **endogenously**

Why Search Frictions?

Several theories of unemployment exist. This chapter focuses on **search frictions**.

Other theories:

- Wage rigidity (minimum wages, unions)
- Efficiency wages (high wages to induce effort)

Search frictions: It takes time, effort, and resources for workers and firms to find each other.

- Workers and jobs are heterogeneous in many dimensions
- Finding a good match is hard
- Some models: explicit matching of heterogeneous workers and jobs
- Other models: matching as a "black box" (reduced-form matching function) — Leading example: **Diamond–Mortensen–Pissarides (DMP) model**

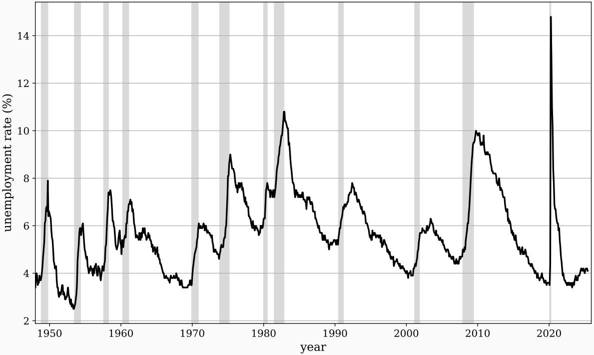
Chapter Roadmap

1. Labor market facts: unemployment, vacancies, Beveridge curve
2. A simple two-state model of unemployment
3. The DMP model: matching, equilibrium, efficiency (Hosios condition)
4. Gross flows across three labor market states
5. The unemployment volatility puzzle (Shimer puzzle)
6. Endogenous separations
7. DMP + neoclassical growth model (consumption, capital)
8. Search with heterogeneous job offers (McCall model)

Labor Market Facts

The US Unemployment Rate

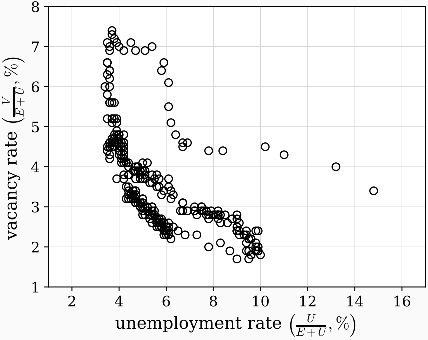
The unemployment rate in the United States



- US civilian non-institutional population (16+) divided into E (employment), U (unemployment), and N (out of labor force)
- Unemployment: searching for a job or on temporary layoff
- Unemployment rate $= U/(E+U)$
- Strongly countercyclical: rises in recessions, falls in expansions

Unemployment and Vacancies

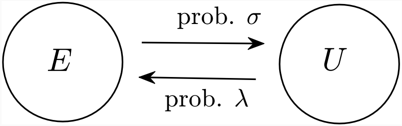
Unemployment and vacancy rates in the United States



- Vacancies are strongly **procyclical** (opposite of unemployment)
- Suggests labor **demand** shifts drive much of cyclical unemployment
- Motivates models that emphasize firms' recruiting decisions

A Simple Model of Unemployment

Setup: Two States



- Workers are either employed (E) or unemployed (U)
- Normalize labor force to 1: $e_t + u_t = 1$
- Ignore flows in/out of labor force
- Unemployment rate equals u_t (since labor force = 1)
- λ: job-finding probability (U to E)
- σ: separation probability (E to U)

Law of Motion and Steady State

Unemployment evolves as:

$$u_{t+1} = (1-\lambda)u_t + \sigma(1-u_t)$$

Steady state ($u_{t+1} = u_t = \bar{u}$):

$$\bar{u} = \frac{\sigma}{\lambda + \sigma}$$

- Decreasing in λ (job finding)
- Increasing in σ (separation)

Convergence:

$$u_{t+1} - \bar{u} = (1-\lambda-\sigma)(u_t - \bar{u})$$

$|1-\lambda-\sigma| \in (0,1) \Rightarrow$ convergence to $\bar{u}$.

Quantitative Example and Duration

Calibration: $\lambda = 0.45$, $\sigma = 0.034$ (monthly)

- $1 - \lambda - \sigma \approx 0.5$: fast convergence
- $\bar{u} \approx 7.0\%$
- From $u_0 = 15\%$: after 6 months, $u \approx 7.2\%$

Average unemployment duration:

$$D = \lambda \cdot 1 + (1-\lambda)\lambda \cdot 2 + (1-\lambda)^2\lambda \cdot 3 + \cdots = \frac{1}{\lambda}$$

Recursive derivation:

$$D_t = \lambda \cdot 1 + (1-\lambda)(1 + D_{t+1})$$

With $D_t = D_{t+1} = D$: $D = 1/\lambda$.

The DMP Model

The Diamond–Mortensen–Pissarides Model

A **search and matching** model (here, a discrete-time version of Pissarides, 1985):

- Workers and firms search for each other
- Matching process summarized by a **matching function**
- Firms post **vacancies** at a cost (costly search, investment)
- Firm and worker split the match surplus via **Nash bargaining**
- Free entry of firms

Key features:

- Only firms engage in costly search (demand side focus)
- Bilateral monopoly once matched $\Rightarrow$ bargaining needed
- Equilibrium may **not** be Pareto efficient (externalities)
- Endogenizes λ from the simple model

The Matching Function

Matches created in period $t+1$:

$$\mathcal{M}_{t+1} = M(u_t, v_t)$$

- Increasing in both arguments, constant returns to scale
- $M(u,v) \leq u$ and $M(u,v) \leq v$
- Random search: all vacancies and workers have equal chance

Labor market tightness: $\theta_t \equiv v_t/u_t$

Job-finding probability (workers): $M(u_t, v_t)/u_t = M(1, \theta_t)$

$$\lambda_w(\theta_t) \equiv M(1, \theta_t), \qquad \text{increasing in } \theta$$

Vacancy-filling probability (firms): $M(u_t, v_t)/v_t = M(1/\theta_t, 1)$

$$\lambda_f(\theta_t) \equiv M(1/\theta_t, 1), \qquad \text{decreasing in } \theta$$

Note: $\lambda_w(\theta_t) = \theta_t \lambda_f(\theta_t)$

Unemployment Dynamics and Steady State

Dynamics of unemployment:

$$u_{t+1} = (1 - \lambda_w(\theta_t))u_t + \sigma(1 - u_t)$$

Steady-state u_t with constant v (call it $\bar{u}$) satisfies:

$$\bar{u} = \frac{\sigma}{\lambda_w(v/\bar{u}) + \sigma}$$

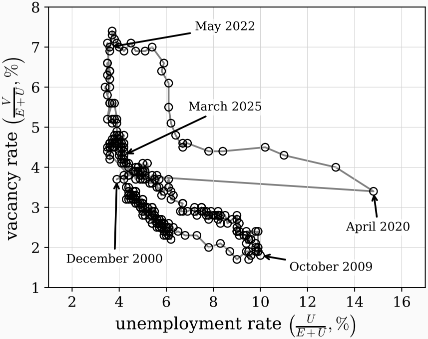
Equivalently:

$$M(v, \bar{u}) + \sigma\bar{u} = \sigma$$

- Unique solution for $\bar{u}$ (RHS constant, LHS increasing)
- Describes a **negative relationship** between v and $\bar{u}$ (LHS increasing in both v and $\bar{u}$) — this is the **Beveridge curve** relationship.

The Beveridge Curve in the Data

Beveridge curve in the United States



- Clear **negative relationship** between unemployment and vacancy rates
- Consistent with the Beveridge curve relationship derived above
- Strong procyclical movement of vacancies supports the demand-side focus

Firm Side: Bellman Equations

Production: one firm + one worker produces z_t units. z_t follows a Markov process.

Matched firm ($w(z)$: wages):

$$J(z) = z - w(z) + \beta\mathbb{E}[(1-\sigma)J(z') + \sigma V(z')]$$

Vacant firm (κ: vacancy cost):

$$V(z) = -\kappa + \beta\mathbb{E}[\lambda_f(\theta)J(z') + (1-\lambda_f(\theta))V(z')]$$

Free entry: $V(z) = 0 \Rightarrow$

$$\frac{\kappa}{\lambda_f(\theta)} = \beta\mathbb{E}[J(z')]$$

Intuition: Cost of creating a vacancy (per match) equals expected discounted value of a filled job.

Worker Side: Bellman Equations

Workers are infinitely lived, risk-neutral, discount factor β:

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t c_t \right]$$

Employed worker:

$$W(z) = w(z) + \beta \mathbb{E}[(1-\sigma)W(z') + \sigma U(z')]$$

Unemployed worker (receives $b < z$):

$$U(z) = b + \beta \mathbb{E}[\lambda_w(\theta) W(z') + (1 - \lambda_w(\theta)) U(z')]$$

b: home production or unemployment insurance.

Generalized Nash Bargaining

Once matched: **bilateral monopoly**. Cannot use the marginal principle.

Flow surplus: $z - b$ (match produces z; if separate, worker gets b, firm nothing). The bargaining below splits the total present value.

Generalized Nash bargaining: split the surplus to maximize weighted product.

$$\max_w \, (\tilde{W}(w, z) - U(z))^\gamma (\tilde{J}(w, z) - V(z))^{1-\gamma}$$

- $\gamma \in (0, 1)$: worker's bargaining power
- $\tilde{W}(w, z), \tilde{J}(w, z)$: values as function of arbitrary wage w

First-order condition on w (surplus sharing rule):

$$(1 - \gamma)(W(z) - U(z)) = \gamma(J(z) - V(z))$$

Equilibrium: Difference Equation in θ_t

Combining the six equilibrium conditions and rearranging:

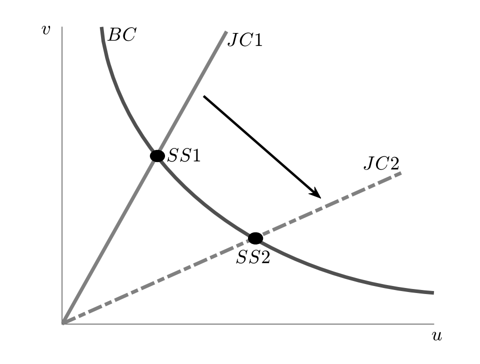
$$\frac{\kappa}{(1-\gamma)\lambda_f(\theta_t)} = \beta\mathbb{E}\left[z_{t+1} - b + \frac{\kappa}{1-\gamma}\left(\frac{1-\sigma-\gamma\lambda_w(\theta_{t+1})}{\lambda_f(\theta_{t+1})}\right)\right]$$

Steady state with constant z (denote $\bar{\theta}$):

$$\frac{\kappa}{(1-\gamma)\lambda_f(\bar{\theta})} = \beta\left[z - b + \frac{\kappa}{1-\gamma}\left(\frac{1-\sigma-\gamma\lambda_w(\bar{\theta})}{\lambda_f(\bar{\theta})}\right)\right]$$

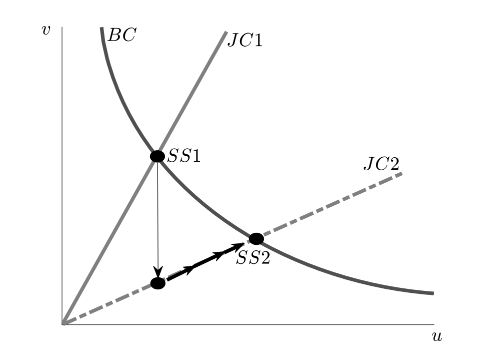
- This is the **job creation condition**
- RHS decreasing in $\bar{\theta} \Rightarrow$ unique solution
- Together with Beveridge curve: pins down $\bar{v}$ and $\bar{u}$

Steady State Determination



- **BC curve**: negative relationship from the Beveridge curve
- **JC line**: straight line through origin with slope $\bar{\theta} = v/u$
- Intersection determines steady state $(\bar{u}, \bar{v})$
- When $z \downarrow$ (recession): $\bar{\theta} \downarrow$, JC rotates down $\Rightarrow \bar{v} \downarrow$, $\bar{u} \uparrow$

Transition Dynamics



Unanticipated, one-time permanent decline in z:

- New steady-state $\bar{\theta}$ solves the new job creation equation
- JC rotates from $JC1$ to $JC2$
- Under a mild condition, θ_t **immediately jumps** to new $\bar{\theta}$
- Since u cannot jump, v drops instantly so that $v/u = \bar{\theta}_{new}$
- Economy then converges gradually along $JC2$ to $SS2$

Efficiency: Private vs. Social Returns

Is the DMP equilibrium efficient? Reduce the equilibrium to two equations.

Define expected match surplus:

$$S_t \equiv \mathbb{E}_t[W(z_{t+1}) - U(z_{t+1}) + J(z_{t+1}) - V(z_{t+1})]$$

In the two-period model, $S_t = \mathbb{E}_t z_{t+1} - b$

Private vacancy posting incentive:

$$\kappa = \lambda_f(\theta_t)(1-\gamma)\beta S_t$$

Social planner's first-order condition in two-period case:

$$\kappa = M_2(u_t, v_t)\beta(\mathbb{E}_t z_{t+1} - b)$$

Thus, for efficiency,

$$M_2(u_t, v_t) = \frac{M(u_t, v_t)}{v_t}(1-\gamma)$$

Two Externalities and the Hosios Condition

Condition for efficiency (two-period version):

$$M_2(u_t, v_t) = \frac{M(u_t, v_t)}{v_t}(1 - \gamma)$$

Two externalities offset:

- **Congestion externality (vacancies)**: Adding a vacancy reduces the chance that other vacancies meet a worker.
- **Bargaining inefficiency**: Firms capture only fraction $1 - \gamma$ of the surplus.

The condition can be rewritten as the **Hosios condition**

$$\eta(u_t, v_t) = \gamma,$$

where

$$1 - \eta(u_t, v_t) \equiv \frac{M_2(u_t, v_t)v_t}{M(u_t, v_t)}$$

η is called the elasticity of matching function.

Infinite Horizon

In the infinite-horizon case (or any model with more than three periods), matching takes away the future opportunity for unemployed workers. This process involves new inefficiencies (at period $t+1$ and beyond):

- **Opportunity cost externality for workers**: A matched worker can no longer search, but also reduces congestion for other workers.
- **Bargaining inefficiency for workers**: Only γ fraction of the worker's opportunity cost is valued in the market.

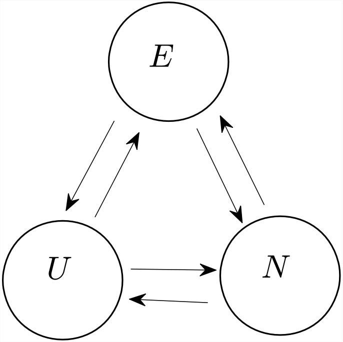
These factors imply **the market surplus S_t may now be different from the social value**. Analogous condition for efficiency

$$\frac{M_1(u_{t+1}, v_{t+1})u_{t+1}}{M(u_{t+1}, v_{t+1})} = \gamma$$

turns out to be identical to the Hosios condition.

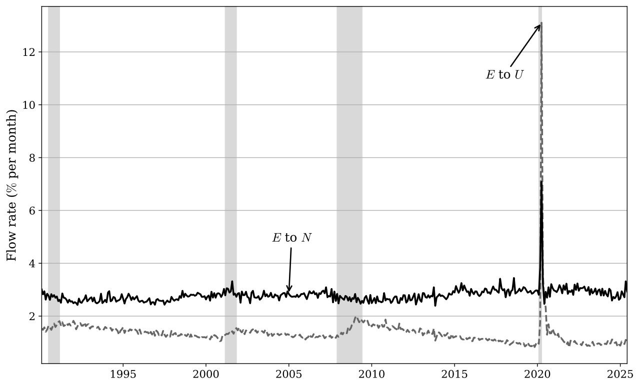
Gross Flows in the Labor Market

Three-State Framework



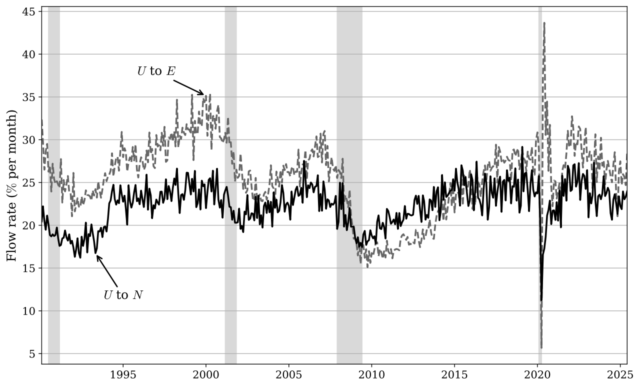
- Previous sections: only E and U
- With three states, there are six gross flows: EU, EN, UE, UN, NE, NU

Flow Rates Out of Employment



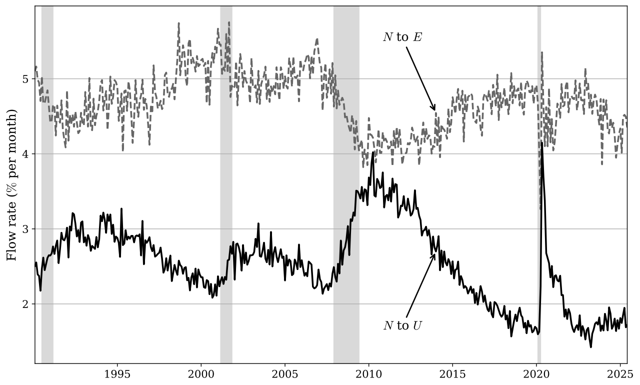
- E-to-U flow rate: **strongly countercyclical** (rises in recessions)
- E-to-N flow rate: more stable
- Countercyclical EU flow: inconsistent with **constant** σ assumption. Motivates endogenous separations

Flow Rates Out of Unemployment



- U-to-E is strongly **procyclical**: consistent with DMP model

Flow Rates Out of Out of the Labor Force



- U increases in recessions because inflows up (EU, NU) and outflows down (UE, UN)

The Unemployment Volatility Puzzle

Functional Forms

Evaluate the quantitative performance of the DMP model.

Cobb-Douglas matching function:

$$M(u, v) = \chi u^{\eta} v^{1-\eta}, \qquad \eta \in (0, 1)$$

Then $\lambda_w(\theta) = \chi\theta^{1-\eta}$ and $\lambda_f(\theta) = \chi\theta^{-\eta}$.

Productivity process (log-deviation $\hat{z}_t = \log z_t - \log \bar{z}$):

$$\hat{z}_{t+1} = \rho\hat{z}_t + \varepsilon_{t+1}, \qquad \varepsilon_{t+1} \sim N(0, \sigma_\varepsilon^2)$$

- ρ: persistence
- σ_ε: innovation standard deviation
- $\log \bar{z}$ normalized to zero

Log-Linearized Solution

Log-linearizing the equilibrium condition around steady state:

$$\mathcal{A}\hat{\theta}_t = \mathbb{E}[\bar{z}\hat{z}_{t+1} + \mathcal{B}\hat{\theta}_{t+1}]$$

where

$$\mathcal{A} = \frac{\kappa\bar{\theta}^\eta \eta}{(1-\gamma)\beta\chi}, \qquad \mathcal{B} = \frac{1-\sigma}{1-\gamma}\frac{\kappa\bar{\theta}^\eta\eta}{\chi} + \frac{\gamma\kappa\bar{\theta}}{1-\gamma}$$

Using method of undetermined coefficients with guess $\hat{\theta}_t = \mathcal{C}\hat{z}_t$:

$$\hat{\theta}_t = (1-\gamma)\left(\frac{\kappa\bar{\theta}^\eta\eta}{\chi}\right)\left[\frac{1}{\rho\beta} - (1-\sigma) + \kappa\gamma\bar{\theta}\right]^{-1}\hat{z}_t$$

Smaller κ, γ, η, larger χ, ρ, β, or smaller $\sigma \Rightarrow$ larger response of θ.

Calibration (Monthly)

Parameter	Value
β (discount factor)	0.996
ρ (persistence)	0.949
σ_ε (shock SD)	0.0065
σ (separation rate)	0.034
χ (matching efficiency)	0.45
b (home production)	0.4
γ (worker bargaining power)	0.72
η (matching elasticity)	0.72

Parameter values. One period = one month.

$\beta = 0.947^{1/12}$ (Cooley and Prescott, 1995); ρ, σ_ε from Hagedorn and Manovskii (2008); $b, \eta, \gamma, \chi, \sigma$ from Shimer (2005) with $\eta = \gamma$ imposing Hosios; κ calibrated so job creation condition holds at $\bar{\theta} = 1$.

US Data: Summary Statistics

	u	v	v/u	z
Standard deviation	0.125	0.139	0.259	0.013
Quarterly autocorrelation	0.870	0.904	0.896	0.765
Correlation matrix				
u	1	-0.919	-0.977	-0.732
v	—	1	0.982	0.460
v/u	—	—	1	0.967
z	—	—	—	1

Summary statistics for quarterly US data.
Monthly data aggregated to quarterly, logged, HP-filtered ($\lambda = 1600$).
Source: Hagedorn and Manovskii (2008).

Model Results and the Shimer Puzzle

	u	v	v/u	z
Standard deviation	0.005	0.016	0.020	0.013
Quarterly autocorrelation	0.826	0.700	0.764	0.765
Correlation matrix				
u	1	-0.839	-0.904	-0.804
v	—	1	0.991	0.972
v/u	—	—	1	0.961
z	—	—	—	1

Model statistics.

- **Correlations: right sign and magnitude**
- **Volatility of u, v, θ: far too small** compared to data
- Known as the **unemployment volatility puzzle** or **Shimer (2005) puzzle**

Why the Response Is So Small

Two intuitive reasons for the small quantitative response:

1. **Benefit of hiring is not very procyclical**
   - Wage rises in booms
   - Dampens the response of profit to a productivity shock
   - Nash bargaining shares productivity gains between firm and worker
2. **Cost of hiring moves with θ**
   - Cost per hire is $\kappa/\lambda_f(\theta)$
   - In booms, θ rises $\Rightarrow \lambda_f(\theta)$ falls $\Rightarrow$ cost per hire rises
   - Dampens the firm's response to a positive productivity shock

Rigid Wages: Model

Many "solutions" to the puzzle have been proposed. Here: **rigid wages** (Hall, 2005; Shimer, 2005) as an example.

Replace Nash bargaining with fixed wage $w = \bar{w}$ (steady-state value). Combining the firm's Bellman equation with free entry:

$$\frac{\kappa}{\lambda_f(\theta_t)} = \beta \mathbb{E}\left[z_{t+1} - \bar{w} + \frac{(1-\sigma)\kappa}{\lambda_f(\theta_{t+1})}\right]$$

Log-linearizing:

$$\hat{\theta}_t = \left(\frac{\kappa \bar{\theta}^{\eta} \eta}{\chi}\right)\left[\frac{1}{\rho\beta} - (1-\sigma)\right]^{-1} \hat{z}_t$$

Two differences vs. Nash case:

- No $(1-\gamma)$ multiplier (firm keeps full gain)
- No $\kappa\gamma\bar{\theta}$ term (no worker outside-option feedback)

Rigid Wages: Results

	u	v	v/u	z
Standard deviation	0.115	0.329	0.425	0.013
Quarterly autocorrelation	0.825	0.693	0.763	0.765
Correlation matrix				
u	1	-0.791	-0.881	-0.784
v	—	1	0.986	0.969
v/u	—	—	1	0.961
z	—	—	—	1

Model statistics with fixed wages.

- Unemployment volatility now **matches the data**
- Wage rigidity substantially amplifies labor market fluctuations
- Sources and magnitude of wage rigidity remain an active research area

Endogenous Separations

Motivation and Setup

Recall: EU flow rate is **strongly countercyclical** in the data. Inconsistent with constant σ.

Extend the model: firm chooses separation probability σ, subject to a cost $c(\sigma)$ with $c'(\sigma) < 0$ (more costly to maintain match at lower σ).

Matched firm's Bellman equation:

$$J(z) = \max_{\sigma} \; z - w(z) - c(\sigma) + \beta \mathbb{E}[(1-\sigma)J(z') + \sigma V(z')]$$

FOC for σ, using $V = 0$ and free entry:

$$-c'(\sigma) = \frac{\kappa}{\lambda_f(\theta)}$$

Optimal σ is a function of z: $\sigma(z)$.

Job Creation with Endogenous Separations

Unemployment dynamics:

$$u_{t+1} = (1 - \lambda_w(\theta_t(z_t)))u_t + \sigma(z_t)(1 - u_t)$$

Job creation condition:

$$\frac{\kappa}{(1-\gamma)\lambda_f(\theta_t)} = \beta \mathbb{E}\left[z_{t+1} - b - c(\sigma(z_{t+1})) + \frac{\kappa}{1-\gamma}\left(\frac{1 - \sigma(z_{t+1}) - \gamma\lambda_w(\theta_{t+1})}{\lambda_f(\theta_{t+1})}\right)\right]$$

- $\sigma(z)$ is determined in equilibrium by the FOC
- This equation determines the dynamics of θ_t
- Since θ depends on z, σ is also a function of z

Log-Linearized System

Maintenance cost: $c(\sigma) = \phi\sigma^{-\xi}$ with $\phi, \xi > 0$. Guess $\hat{\theta}_t = \mathcal{G}\hat{z}_t$.

FOC and cost log-linearize to:

$$\hat{\sigma}(z_t) = -\frac{\eta}{\xi + 1}\mathcal{G}\hat{z}_t, \qquad \hat{c}(z_t) = \frac{\xi\eta}{\xi + 1}\mathcal{G}\hat{z}_t$$

After substitution, $G = \Theta/\Gamma$ where

$$\Theta = (1-\gamma)\left(\frac{\kappa\bar{\theta}^{\eta}\eta}{\chi}\right)\left[\frac{1}{\rho\beta} - (1-\bar{\sigma}) + \kappa\gamma\bar{\theta}\right]^{-1}\bar{z}$$

$$\Gamma = 1+(1-\gamma)\left(\frac{\kappa\bar{\theta}^{\eta}\eta}{\chi}\right)\left[\frac{1}{\rho\beta} - (1-\bar{\sigma}) + \kappa\gamma\bar{\theta}\right]^{-1}\left(\frac{\xi\eta}{\xi+1}\right)\left(1 - \frac{1}{1-\gamma}\right)\bar{c}$$

Results with Endogenous σ

Calibration: $\sigma, \rho, \sigma_\varepsilon, \chi, b, \gamma, \eta$ as before. Set $\bar{\sigma} = 0.034$.

Using std$(\hat{\lambda}_w)$/std$(\hat{\sigma}) = (1-\eta)(1+\xi)/\eta \approx 1$ (Krusell et al., 2017), and $\eta = 0.72$: $\xi = 1.6$.

	u	v	v/u	z
Standard deviation	0.010	0.011	0.021	0.013
Quarterly autocorrelation	0.862	0.623	0.764	0.765

Model statistics with endogenous σ.

- Volatility of u slightly larger than constant-σ case
- Far too small fluctuations in u and v compared to data

Endogenous σ + Rigid Wages

With rigid wages, equilibrium condition becomes:

$$\frac{\kappa}{\lambda_f(\theta_t)} = \beta\mathbb{E}\left[z_{t+1} - \bar{w} - c(z_{t+1}) + \frac{(1-\sigma(z_{t+1}))\kappa}{\lambda_f(\theta_{t+1})}\right]$$

Log-linearization: endogenous c and σ terms **exactly cancel**.

Result identical to constant-σ fixed-wage case.

	u	v	v/u	z
Standard deviation	0.217	0.232	0.433	0.013
Quarterly autocorrelation	0.852	0.609	0.763	0.765

Model statistics with endogenous σ and fixed wages.

- Volatility of u even **larger than data**
- Less extreme wage rigidity would still generate substantial fluctuations

DMP in the Neoclassical Growth Model

Motivation: Connecting to the Workhorse Model

Connect DMP to the **neoclassical growth model**. Two important changes:

1. Concave utility with explicit consumption-saving decision
2. Capital as input in production (in addition to labor)

Closed economy: profit income from firm ownership is made explicit.

Follows Krusell, Mukoyama, and Şahin (2010). Earlier work: Merz (1995), Andolfatto (1996).

Family Structure

- Unit square of consumers indexed by (i, j)
- i: family index; j: member index within family
- Families are identical $\Rightarrow$ use the representative family
- **Full insurance within family**: income pooled, all members consume same amount
- Some members employed, others unemployed—doesn't matter for individual consumption

Utility of representative member:

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \mathbf{U}(c_t) \right]$$

with $\mathbf{U}'(\cdot) > 0$, $\mathbf{U}''(\cdot) < 0$.

Family budget constraint (d_t: dividend):

$$c_t + k_{t+1} = (1 + r_t - \delta)k_t + (1 - u_t)w_t + u_t b + d_t$$

Production and Capital

Each match uses capital k_f and one unit of labor to produce $z_t k_f^\alpha$, $\alpha \in (0,1)$.

Firms rent capital each period from families:

$$\max_{k_f}\; z_t k_f^\alpha - r_t k_f \quad \Rightarrow \quad \alpha z_t k_f^{\alpha-1} = r_t$$

In equilibrium, $1-u_t$ matches divide the aggregate capital K_t:

$$r(z_t, K_t, u_t) = \alpha z_t \left(\frac{K_t}{1-u_t}\right)^{\alpha-1}$$

Let $X_t \equiv (z_t, K_t, u_t)$. Surplus per match:

$$y(X_t) = (1-\alpha) z_t \left(\frac{K_t}{1-u_t}\right)^{\alpha}$$

Consumption-Saving Block

Family's Bellman equation:

$$\mathbf{V}(k, X) = \max_{c,k'} \mathbf{U}(c) + \beta\mathbb{E}[\mathbf{V}(k', X')|z]$$

subject to

$$c + k' = (1 + r(X) - \delta)k + (1 - u)w(X) + ub + d(X)$$

$$K' = \Omega(X), \qquad u' = (1 - \lambda_w(\theta(X)))u + \sigma(1 - u)$$

Family takes $r(X), w(X), d(X), \Omega(X), \theta(X)$ as given.

From decision rules: aggregate consumption $C(X) = c(K, X)$;
next-period capital $\Omega(X) = k'(K, X)$.

The State Price

With concave utility, the discount factor is not constant. The state price (price of Arrow security) is:

$$Q(z', X) = \beta f(z'|z) \frac{\mathbf{U}'(C(z', \Omega(X), u'(X)))}{\mathbf{U}'(C(X))}$$

- $f(z'|z)$: transition density
- Discounting future payoffs: use $Q(z', X)$ instead of β
- Firms value future profits according to the stochastic discount factor

Labor Market Block

Matched firm:

$$J(X) = y(X) - w(X) + \int Q(z', X)[(1-\sigma)J(X') + \sigma V(X')]dz'$$

Vacant firm (with free entry $V(X) = 0$):

$$\frac{\kappa}{\lambda_f(\theta(X))} = \int Q(z', X)J(X')dz'$$

Employed and unemployed worker values:

$$W(X) = w(X) + \int Q(z', X)[(1-\sigma)W(X') + \sigma U(X')]dz'$$

$$U(X) = b + \int Q(z', X)[\lambda_w(\theta(X))W(X') + (1-\lambda_w(\theta(X)))U(X')]dz'$$

Nash bargaining: $(1-\gamma)(W(X) - U(X)) = \gamma(J(X) - V(X))$

Job Creation Condition, Wage, and Dividend

Job creation condition:

$$\frac{\kappa}{(1-\gamma)\lambda_f(\theta(X))} = \int Q(z', X)\left[y(X') - b + \frac{\kappa}{1-\gamma}\left(\frac{1-\sigma-\gamma\lambda_w(\theta(X'))}{\lambda_f(\theta(X'))}\right)\right] dz'$$

Wage (from Nash bargaining):

$$w(X) = \gamma(y(X) - b) + b + \gamma\theta(X)\kappa$$

Dividend (all firms' profits minus vacancy cost):

$$d(X) = (1-u)(y(X) - w(X)) - \kappa\theta(X)u$$

Algorithm: [Alternative 1]: log-linearize. [Alternative 2] guess Q, solve labor block for θ, w, d, solve consumption-saving block, update Q, iterate.

Calibration and Labor Market Results

Calibration:

- $\mathbf{U}(c) = \log(c)$
- $\alpha = 0.4$ (Cooley and Prescott, 1995)
- $\delta = 0.004$ monthly (= 0.012 quarterly)
- b adjusted to 0.4 times steady-state $y(X)$

	u	v	v/u	z
Standard deviation	0.005	0.017	0.022	0.015
Quarterly autocorrelation	0.819	0.688	0.755	0.763
corr$(\cdot, z)$	0.089	-0.071	-0.078	1

Model statistics with generalized Nash bargaining: labor market.

- The Shimer puzzle remains
- Correlations of u, v, and v/u with z near zero: $y(X)$ depends on K and u too, not only z

Business Cycle Statistics

	Y	C	I	L	Y/L
Standard deviation	0.014	0.003	0.059	0.0004	0.014
Correlation with Y	1	0.875	0.991	0.902	0.99992

Business cycle statistics with generalized Nash bargaining.

- Standard RBC-like pattern: $C, I, L, Y/L$ procyclical
- I more volatile than Y
- C less volatile than Y
- L fluctuates **much less** than Y: reflects unemployment volatility puzzle

Rigid Wages with Capital

Assume wage is rigid but bounded below by flow output:

$$\tilde{w}(X) = \max\{\bar{w}, y(X)\}$$

Max is needed because $y(X)$ moves with K and u, so profit could otherwise go negative. In most periods, $\tilde{w}(X) = \bar{w}$.

Job creation condition:

$$\frac{\kappa}{\lambda_f(\theta(X))} = \int Q(z', X)\left[y(X') - \tilde{w}(X') + \frac{(1-\sigma)\kappa}{\lambda_f(\theta(X'))}\right] dz'$$

Dividend: $d(X) = (1-u)(y(X) - \tilde{w}(X)) - \kappa\theta(X)u$

Rigid Wages: Results

	u	v	v/u	z
Standard deviation	0.083	0.269	0.339	0.015
Quarterly autocorrelation	0.818	0.671	0.744	0.763

Model statistics with rigid wages: labor market.

	Y	C	I	L	Y/L
Standard deviation	0.017	0.003	0.060	0.007	0.011
Correlation with Y	1	0.792	0.989	0.898	0.964

Business cycle statistics with rigid wages.

- Much larger response of v and u to shocks
- L standard deviation one order of magnitude larger than Nash case

Heterogeneous Jobs and the McCall Search Model

Motivation: Wage Dispersion

Previously: homogeneous jobs, all matches accepted. Now: heterogeneous job offers.

Diamond paradox (Diamond, 1971): In a search model with homogeneous firms, costly search, and on-the-job search unavailable, all firms offer the **worker's reservation wage** (the minimum wage acceptable).

- Nash equilibrium even with tiny search costs
- Unique: any firm with higher wage would want to cut

A workaround (there can be other ways): Assume **exogenous** heterogeneity in wage offers. Can be justified by heterogeneous jobs or on-the-job search.

This is the **McCall (1970) search model**: worker chooses whether to accept or keep searching.

McCall Model: Setup

Focus on the worker side. Worker is infinitely lived, risk-neutral:

$$\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t c_t\right]$$

- Unemployed worker earns $b > 0$ (home production/UI)
- Each period: receives a wage offer $w \sim F(w)$, support $[0, w_u]$
- Employed worker faces separation probability $\sigma \in (0,1)$

Bellman equation (unemployed):

$$U = b + \beta \int_0^{w_u} \max\{W(w), U\} dF(w)$$

Bellman equation (employed at wage w):

$$W(w) = w + \beta[(1-\sigma)W(w) + \sigma U]$$

The Reservation Wage

Solving the employed worker's equation:

$$W(w) = \frac{w + \beta\sigma U}{1 - \beta(1-\sigma)}$$

$W(w)$ increasing in w. There exists a **reservation wage** w^* such that:

$$W(w^*) = U$$

Worker accepts if and only if $w \geq w^*$.

Since $W(w^*) = U$, the expression $(1-\beta)U = w^*$ follows, and:

$$w^* = b + \frac{\beta}{1 - \beta(1-\sigma)} \int_{w^*}^{w_u} (w - w^*) dF(w)$$

Job-finding probability: $\lambda = 1 - F(w^*)$ (decreasing in w^*; choosier $\Rightarrow$ less frequent).

Comparative Statics

Rewrite equation as $\mathbf{G}(w^*, \beta, \sigma, b) = 0$ where

$$\mathbf{G}(w^*, \beta, \sigma, b) = w^* - b - \frac{\beta}{1 - \beta(1 - \sigma)} \int_{w^*}^{w_u} (w - w^*) dF(w)$$

Partial derivatives: $\partial \mathbf{G} / \partial \beta < 0$, $\partial \mathbf{G} / \partial \sigma > 0$, $\partial \mathbf{G} / \partial b < 0$, $\partial \mathbf{G} / \partial w^* > 0$.

By the implicit function theorem, w^* is:

- **Increasing in** β: more patient $\Rightarrow$ higher weight on future gains
- **Increasing in** b: better outside option $\Rightarrow$ choosier
- **Decreasing in** σ: short match duration $\Rightarrow$ less worth waiting

Effect of a Change in the Average Wage

Replace wage offer w with $w + \varepsilon$. How does w^* change with ε (at $\varepsilon = 0$)?

Define

$$\mathbf{G}(w^*, \varepsilon) \equiv w^* + \varepsilon - b - \frac{\beta}{1 - \beta(1-\sigma)} \int_{w^*}^{w_u} [(w+\varepsilon) - (w^* + \varepsilon)] dF(w)$$

After computing partial derivatives (Leibniz's rule):

$$\frac{d(w^* + \varepsilon)}{d\varepsilon} = \frac{\beta[1 - F(w^*)]}{1 - \beta(1-\sigma) + \beta[1 - F(w^*)]} \in (0, 1)$$

Interpretation:

- Average wage up by \$1 $\Rightarrow$ reservation wage up by *less* than \$1
- Because b is fixed, unemployment becomes relatively less attractive
- Workers become **less choosy** in relative terms

Mean-Preserving Spreads

How does an increase in **dispersion** of wage offers affect w^*?

Mean-preserving spread:

- Given x with distribution F, construct $\tilde{x} = x + z$ where $\mathbb{E}[z] = 0$
- The new distribution $G(\cdot)$ is a mean-preserving spread of $F(\cdot)$

Equivalent characterization: G is a mean-preserving spread of F iff

$$\int_0^x G(t)dt \geq \int_0^x F(t)dt \quad \text{for all } x$$

Result: An increase in wage dispersion **raises** w^*.

- More dispersion $\Rightarrow$ higher option value of waiting
- Workers reject more offers hoping for a better one
- Classic result in the search literature

Frictional Wage Dispersion

Without search frictions, all workers work with w^u. With search frictions, there is some **frictional wage dispersion**

Define the mean wage as $w^M \equiv [\int_{w^*}^{w^u} wF(w)]/(1 - F(w^*))$ and the replacement rate as $\rho \equiv b/w^M$

Also define the **mean-min ratio** of the accepted wage as

$$Mm \equiv \frac{w^M}{w^*}$$

It can be shown that, in the above McCall model,

$$Mm = \frac{1 + \beta\lambda/(1 - \beta(1 - \sigma))}{\rho + \beta\lambda/(1 - \beta(1 - \sigma))}$$

With a reasonable calibration, Mm is very close to 1 (when $\beta = 0.996$, $\sigma = 0.034$, $\lambda = 0.45$, and $b = 0.4$, $Mm = 1.031$). This is called the **frictional wage dispersion puzzle**. Potential solutions: high cost of unemployment, on-the-job search

Summary

Key Takeaways (I)

1. **Labor market facts**: unemployment countercyclical, vacancies procyclical, Beveridge curve, large gross flows
2. **Simple model**: $\bar{u} = \sigma/(\lambda + \sigma)$; average duration $= 1/\lambda$
3. **DMP model**: matching function, Nash bargaining, free entry; Beveridge curve + job creation condition
4. **Efficiency**: Hosios condition $\eta = \gamma$ makes equilibrium efficient (congestion externalities offset by bargaining inefficiency)

Key Takeaways (II)

5. **Shimer puzzle**: Standard DMP generates too little unemployment volatility relative to data
6. **Rigid wages** (Hall, 2005; Shimer, 2005): substantially amplify labor market fluctuations
7. **Endogenous separations**: capture countercyclical EU flow; with rigid wages, match data
8. **DMP + NGM**: stochastic discount factor, closed economy; puzzle persists under Nash, resolved by rigid wages
9. **McCall model**: reservation wage w^*; increasing in b, β, decreasing in σ; dispersion $\uparrow$ raises w^*; analysis of frictional wage dispersion