Patterns of Specialization*

Toshihiko Mukoyama
Department of Economics
Concordia University
and
CIREQ
mukoyama@alcor.concordia.ca

Ayşegül Şahin
Federal Reserve Bank of New York
aysegul.sahin@ny.frb.org

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Abstract

This paper constructs a matching model with heterogeneous workers. A worker has two-dimensional talent, and decides to specialize in one of two tasks taking his partner's talent into account. The patterns of specialization under matching friction are analyzed. Due to matching friction, there may be mismatch of talents, and one may be forced to specialize in a task in which he is not good at. In the example we present, the workers are divided into groups. Some workers accept to match with any other workers, and some workers only match with workers from outside their group. In equilibrium, low-skilled workers tend to have a high unemployment rate. It is shown that the aggregate output can increase by more generous unemployment insurance.

Keywords: matching, specialization, unemployment insurance
JEL Classifications: E24, J41, J64, J65

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1 Introduction

Economists have long been emphasizing the social benefit of having people specialize in what they are good at. In Plato's *Republic*, Socrates argues:

> it occurred to me that, in the first place, no two of us are born exactly alike. We have different natural aptitudes, which fit us for different jobs. ... Quantity and quality are therefore more easily produced when a man specializes appropriately on a single job for which he is naturally fitted, and neglect all others. (370, pp. 56–57)

In a modern society, a large part of production is conducted in teams. In team production, specialization requires finding a good partner. When one specializes in one task, he has to find someone else who specializes in the tasks that he left out. Very often, finding a good partner who complements one's ability is not easy. If one cannot find a good partner, he might be forced to carry out a task that he is not good at. Through the channel of frictions in searching for a partner, significant inefficiency may result.

This paper develops an equilibrium matching model to analyze this issue. Under matching friction, how often do people specialize in the tasks they are not good at? How does the matching friction affect the pattern of specialization in the entire economy? How is the aggregate output of the economy affected by this channel? How do policies change the patterns of specialization? What is the resulting income distribution? We analyze these questions.[1]

This paper provides a natural framework to analyze "mismatch" of talents under heterogeneous agents and matching frictions. In our model, mismatch of talents can result in an inefficient outcome through two mechanisms. First, mismatch leads to unemployment if two workers decide not to form a match because their talents are not compatible with each

[1] There is a recent literature in search-theoretic models of money which investigates the issue of specialization. These papers consider search frictions in trading, rather than forming a long-run relationship as in the current paper. See, for example, Kiyotaki and Wright (1993) and Camera, Reed, and Waller (2003).

other. Second, mismatch leads to inefficient specialization if two workers decide to form a match even though their talents are not compatible with each other. One striking policy implication of our model is that the aggregate output can increase by more generous unemployment insurance. More generous unemployment insurance makes workers more selective and reduces the second inefficiency due to mismatch.

In labor economics, there is a long tradition of analyzing specialization patterns using the Roy model (Roy [1951]).[2] In this literature, it is commonly assumed that workers are price-takers and choose their tasks (or sectors) based on their talent and the market wage. Sattinger (2003) extends the standard single-agent search model and allows the workers to choose which sector to search in *prior to* engaging in search. In our model, search activity is entirely random, and the workers can decide whether to accept or reject the match depending on the matched partner's characteristics *after* the match is formed. Another important difference of our work from Sattinger's model is that our model is an equilibrium model, and a worker's decision influences the matching environment of the others.

There is a large literature of two-sided matching with heterogeneous agents.[3] Most of the literature, however, concerns one-dimensional heterogeneity across agents. In those models, agents can be ranked from skilled to unskilled (or attractive to unattractive), and the resulting matching patterns are analyzed in various environments. One-dimensional models, however, are not able to analyze the issue of specialization. If one is equally good at carrying out all tasks and another is equally bad at all tasks, the efficiency of the economy is hardly affected by the arrangement of who is assigned to which task. This paper extends the model to the case where the agents have two-dimensional talents. Matching friction induces some agents to specialize in the tasks that they are not good at.

Sundaram (2003) extends Burdett and Coles (1997) model to allow for two-dimensional characteristics ("charm" and "taste") on one side of the match. Her main concern is how in-

[2] See, for example, the survey by Sattinger (1993).

[3] See, for example, McNamara and Collins (1990), Lu and McAfee (1996), Burdett and Coles (1997), Eeckhout (1999), Bloch and Ryder (2000), Shimer and Smith (2000), Shi (2001), Smith (2002), Delacroix (2003), Danthine (2004), Shimer (2004).

dividual frequency of match is related to search friction. Our model permits two-dimensional characteristics on both sides. We analyze the aggregate consequences, as well as the individual behavior.

This paper is organized as follows. The next section describes the model. In Section 3, we characterize an example with a specific distribution. Section 4 discusses certain aspects of the results in Section 3. Section 5 concludes.

2 Model

There are continuum of workers with measure 1. A worker can produce when matched with another worker.[4] Here, "matching" describes a situation where they meet and both accept each other as a production partner. For production, there are two tasks to perform: X and Y. One worker specializes in task X, and the other specializes in task Y. Workers differ in their talents to perform each task. Specifically, let the vector $(x_i, y_i) \in [0,1] \times [0,1]$ represent worker i's talent to perform task (X, Y). Assume that the production function is linear:

$$\Pi(i,j) = 2(x_i + y_j)$$

when worker i specializes in task X and worker j specializes in task Y. Assume that workers divide the output in half,[5] therefore one worker receives $w(i,j) \equiv x_i + y_j$. Clearly both workers agree upon the efficient pattern of specialization *within* the match, and i specializes in X if

$$x_i + y_j \geq x_j + y_i.$$

Figure 1 is drawn from the viewpoint of worker i. If he matches with a worker whose characteristics fall on region A, worker i will specialize in task X. If his partner's characteristics fall on region B, worker i will specialize in task Y. Since his x_i is low, it can easily be guessed

[4]As in the standard "partnership" models, we can interpret this matching process in two ways. Here we interpret that two workers are drawn from the same pool of workers. Instead, it is possible to interpret (as in the literature cited in footnote 3) that there are two symmetric groups of workers, and one is drawn from each group.

[5]An alternative assumption would be to share the total surplus by, for example, Nash bargaining.

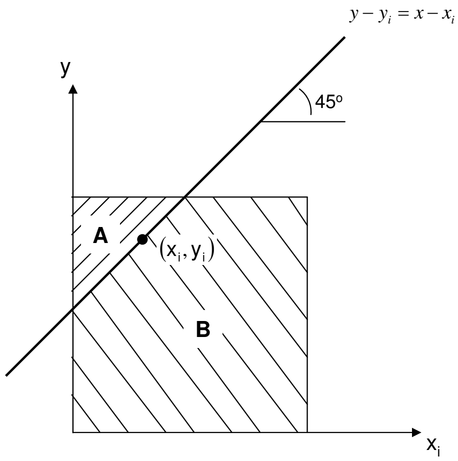

Figure 1: Efficient pattern of specialization within match

that it is not good for economy-wide efficiency to make him match with a region A worker. The following proposition confirms this intuition.

The following characterizes an *economy-wide* efficient (maximizing total output) pattern of match.

Proposition 1 *When the match distribution is symmetric with respect to $y = x$, it is necessary for efficiency that (except for measure zero matches) a worker i with $x_i > y_i$ is matched with a worker j with $x_j < y_j$ and vice versa.*

Proof: See Appendix. □

An immediate corollary is that for efficiency, a worker i with $x_i > y_i$ always specializes in task X and a worker j with $y_j > x_j$ always specializes in task Y (See Figure 2). In fact, this will be the pattern which will realize in the frictionless Roy model. Is this pattern of specialization realized when there are matching frictions? The answer is no—under matching frictions, there are workers who specialize in tasks that they are not good at. To proceed with the analysis, we formally specify the environment below.

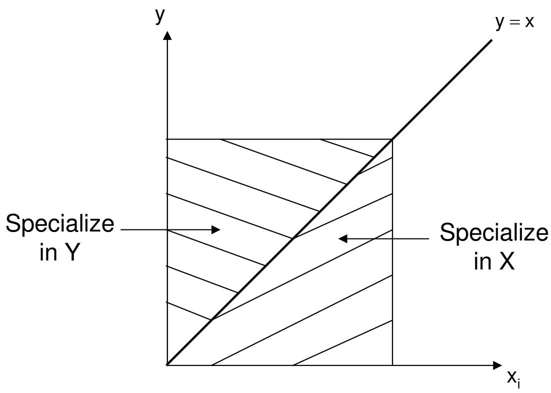

Figure 2: Economy-wide efficient pattern of specialization

Suppose that an unmatched worker receives hdt amount of utility during the time period of dt. The value of h can be interpreted as the level of unemployment insurance. Here, we do not explicitly consider how h is financed. Appendix A shows that when h is financed by a lump-sum tax, the model with tax is equivalent to the current model. Assume that an unmatched worker randomly meets with another unmatched worker with the probability αdt during the time period of dt.[6] Let $\mathcal{F}_i$ be the *feasible set* for i: the set of workers who accepts i if they meet. Let $\mathcal{F}_i^c$ be the complementary set of $\mathcal{F}_i$. The Bellman equation for an unmatched worker i is:

$$U_i = hdt + \frac{1}{1+rdt}\left[\alpha dt\left(\int_{\mathcal{F}_i}\max\langle V_i(x_j,y_j),U_i\rangle dF(x_j,y_j) + \int_{\mathcal{F}_i^c} U_i dF(x_j,y_j)\right) + (1-\alpha dt)U_i\right],$$

where r is the discount rate, $F(x_j, y_j)$ is the distribution function of (x_j, y_j) among the unemployed workers, U_i is the value of i being unemployed, and $V_i(x_j, y_j)$ is the value of i being matched with worker j.

Assume that a match is subject to exogenous separation with probability δdt during the time period dt. The Bellman equation for a worker i matched with worker j is:

$$V_i(x_j,y_j) = \max\langle x_i+y_j, x_j+y_i\rangle dt + \frac{1}{1+rdt}\left[(1-\delta dt)V_i(x_j,y_j) + \delta dt U_i\right].$$

[6] This can be thought as a constant-returns-to-scale matching technology.

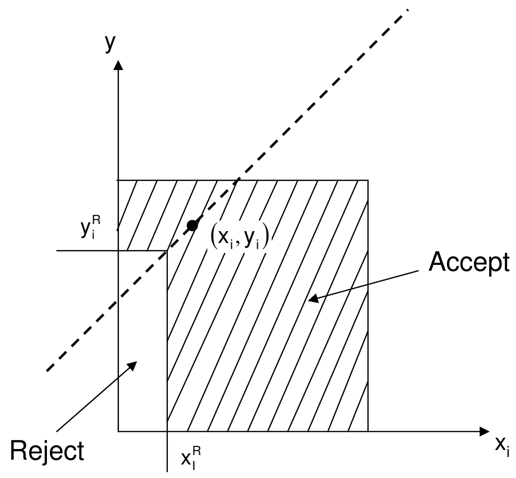

Figure 3: Decision rule

Note that from the Contraction Mapping Theorem (Stokey and Lucas with Prescott [1989] Theorem 3.2 and Corollary 1), U_i is nondecreasing in x_i and y_i, and $V_i(x_j, y_j)$ is nondecreasing in x_i, y_i, x_j, and y_j when $\mathcal{F}_i$ is monotonic ($\mathcal{F}_i \supseteq \mathcal{F}_{i'}$ when $x_i \geq x_{i'}$ with $y_i = y_{i'}$ or $y_i \geq y_{i'}$ with $x_i = x_{i'}$). Taking $dt \to 0$,

$$rU_i = h + \alpha \int_{\mathcal{F}_i} \max\langle V_i(x_j, y_j) - U_i, 0\rangle dF(x_j, y_j) \tag{1}$$

and

$$rV_i(x_j, y_j) = \max\langle x_i + y_j, x_j + y_i\rangle - \delta[V_i(x_j, y_j) - U_i] \tag{2}$$

hold.

As in the standard models of search and matching, the decision of accepting a match or not exhibits a "reservation value" property. What is new here is that the reservation value is not one-dimensional. In particular, the reservation value depends on the specialization pattern. When worker i meets a worker j in region A of Figure 1, for i the value of x_j is not relevant since j will always specialize in task Y in this match. Therefore, i will set only set the reservation value on y_j (call this y_i^R) and accept the match if and only if $y_j \geq y_i^R$. Similarly, when i meets a worker j in region B, he sets a reservation value on x_j and accepts

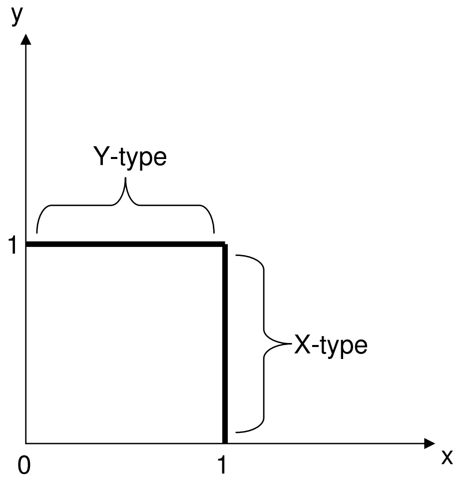

Figure 4: X-type and Y-type

a match if and only if $x_j \geq x_i^R$. The decision rule is depicted in Figure 3. Note that (x_i^R, y_i^R) lies on the 45-degree line that goes through (x_i, y_i), but it can be larger or smaller than (x_i, y_i).

3 Example: "Flipped-L" Distribution Case

It is difficult to make much progress without imposing some restrictions on the distribution of talent. In this section, we assume a specific distribution of talent, and characterize the equilibrium.

3.1 Talent Distribution

Here we consider a "flipped-L" shape distribution of talent (x_i, y_i). (See Figure 4.) There are two types of agents. The first type, called X-type, has a skill level $x_i = 1$ in task X and a skill level $y_i \in [0, 1]$ in task Y. The second type, called Y-type, has a skill level $y_i = 1$ in task Y and a skill level $x_i \in [0, 1]$ in task X. We assume that the distribution of y_i across X-type agents and the distribution of x_i across Y-type agents are symmetric and follow a continuous distribution. Note that from Proposition 1, the efficient matching pattern is realized if each

X-type worker matches with a Y-type worker. The meeting of the same type workers (X-type with X-type and Y-type with Y-type) can be considered as "mismatch": if they don't form a match, it results in unemployment due to mismatch. If they form a match, one of the workers has to specialize in a task that he is not good at. In both cases, the mismatch creates inefficiency.

3.2 Nash Equilibrium

First, we construct an equilibrium *given* the talent distribution of the unemployed: $F(x, y)$. This is the "Nash equilibrium" problem in Burdett and Coles (1999) terminology. Clearly, $F(x, y)$ is endogenous and determined by the behavior of the workers. We will endogenize $F(x, y)$ in the following section.

We impose some restrictions on the parameters.

Assumption 1 *Assume that* $2 > h > \underline{h}$, *where* $\underline{h} \equiv 1 - \alpha/2(r + \delta)$.

Clearly, when h is too high, no one want to accept the match. When h is too low, everyone will accept any match. We exclude these equilibria by this assumption.

In this paper, we focus on a stationary equilibrium that is symmetric across type X and type Y. We assume that $F(x, y)$ is symmetric with respect to the $x = y$ line and strictly positive everywhere along the "flipped-L" edge. Without loss of generality, we will focus on a Y-type worker. For the sake of brevity, we index the workers by the value of x hereafter. With a slight abuse of notation, we let $\mathcal{F}(x_i) \subseteq [0, 1]$ represent the values of x of the workers who would accept the worker i. Since $h < 2$, all the Y-type workers accept to match with any of the X-type workers, that is, $1 \in \mathcal{F}(x_i)$ for any $x_i < 1$. When a Y-type worker i and an X-type worker j match, j (whose $x = 1$) always specialize in X and value of y_j does not matter for production. When a Y-type worker i and another Y-type worker j match, both have $y = 1$. Therefore, when we consider the matching decision of Y-type workers, we do not need to keep track of the partner's value of y. Therefore, for the sake of brevity, we will write $V_i(x_j, y_j)$ as $V_i(x_j)$ and $F(x_j, y_j)$ as $F(x_j)$ hereafter. We consider the case where $F(x_j)$

is continuous for $x_j \in [0, 1)$.[7] (Clearly, it is not continuous at $x_j = 1$, where all the X-type workers are.) The following lemma is straightforward from the fact that everyone prefers to match with a person with higher x.

Lemma 1 *When $x' \geq x$, $\mathcal{F}(x') \supseteq \mathcal{F}(x)$.*

Proof: Suppose that $x_i \in \mathcal{F}(x)$, that is, i accepts to produce with a worker with X-skill x. In other words, $V_i(x) \geq U_i$. It suffices to show that $i \in \mathcal{F}(x')$ for any $x' \geq x$. From (2), $V_i(x') \geq V_i(x)$. Therefore, when $V_i(x) \geq U_i$, $V_i(x') \geq U_i$ also holds. □

This Lemma ensures that $V_i(x)$ and U_i are monotonically increasing in x_i. Define an *acceptance set* $\mathcal{A}(x_i) \subseteq [0, 1]$ as the set of the x-values of workers whom i is willing to accept. With a similar argument as Lemma 1, we can establish the following.

Lemma 2 $\mathcal{A}(x)$ *is either empty or an interval containing 1.*

Proof: Suppose that $x \in \mathcal{A}(x_i)$. That is, $V_i(x) \geq U_i$. Then, from the previous argument, $V_i(x') \geq U_i$ also holds for any $x' \geq x$. □

In fact, Lemma 2 is a direct consequence of Lemma 1, considering the fact that $\mathcal{A}(x)$ is the inverse correspondence of $\mathcal{F}(x)$. Define the *matching set* $\mathcal{M}(x_i) \equiv \mathcal{F}(x_i) \cap \mathcal{A}(x_i)$. This is the set of x-values of workers that i actually matches with once he meets. Utilizing this notation, (1) can be rewritten as:

$$rU_i = h + \alpha \int_{\mathcal{M}(x_i)} [V_i(x_j) - U_i] dF(x_j). \tag{3}$$

To characterize the acceptance set, first note that given U_i, $V_i(x)$ is continuous and increasing in x. In particular, from (2),

$$V_i(x) = \begin{cases} (x + 1 + \delta U_i)/(r + \delta) & \text{if } x \geq x_i, \\ (x_i + 1 + \delta U_i)/(r + \delta) & \text{if } x < x_i. \end{cases}$$

[7] This will be verified later when we analyze endogenous $F(x_j)$.

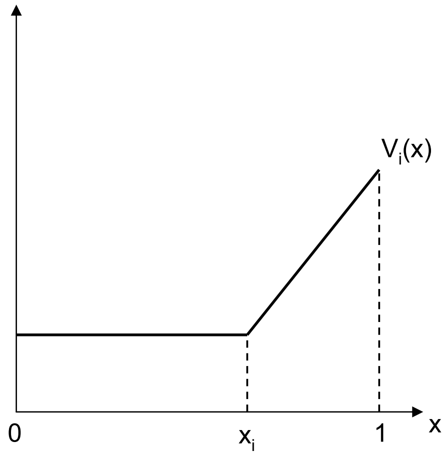

Figure 5: $V_i(x)$

See Figure 5. Note that $V_i(x)$ is (weakly) increasing in x_i, given U_i. The decision rule for accepting or not depends on whether $V_i(x) \geq U_i$. $V_i(x) - U_i$ can be expressed as:

$$V_i(x) - U_i = \begin{cases} (x + 1 - rU_i)/(r + \delta) & \text{if } x \geq x_i, \\ (x_i + 1 - rU_i)/(r + \delta) & \text{if } x < x_i. \end{cases} \tag{4}$$

From Lemma 2, the acceptance set $\mathcal{A}(x_i)$ can be characterized by its lower bound. Denote it by $\underline{x}_i$. From (4),

$$\underline{x}_i = \begin{cases} 0 & \text{if } x_i + 1 - rU_i \geq 0, \\ rU_i - 1 & \text{otherwise.} \end{cases} \tag{5}$$

See Figure 6.

Now we will prove the existence of the equilibrium that exhibits a "group" structure.

Theorem 1 (Existence and Uniqueness of the Nash Equilibrium)

There exists a unique Nash equilibrium of this economy and has the following property. The agents are divided into two groups within each types. Y-type can be divided into Group $Y1$ and Group $Y2$.

- $Y1$*:* $x_i \in [\hat{x}, 1]$. $\mathcal{M}_i (= \mathcal{F}_i = \mathcal{A}_i)$ *is the entire population.*

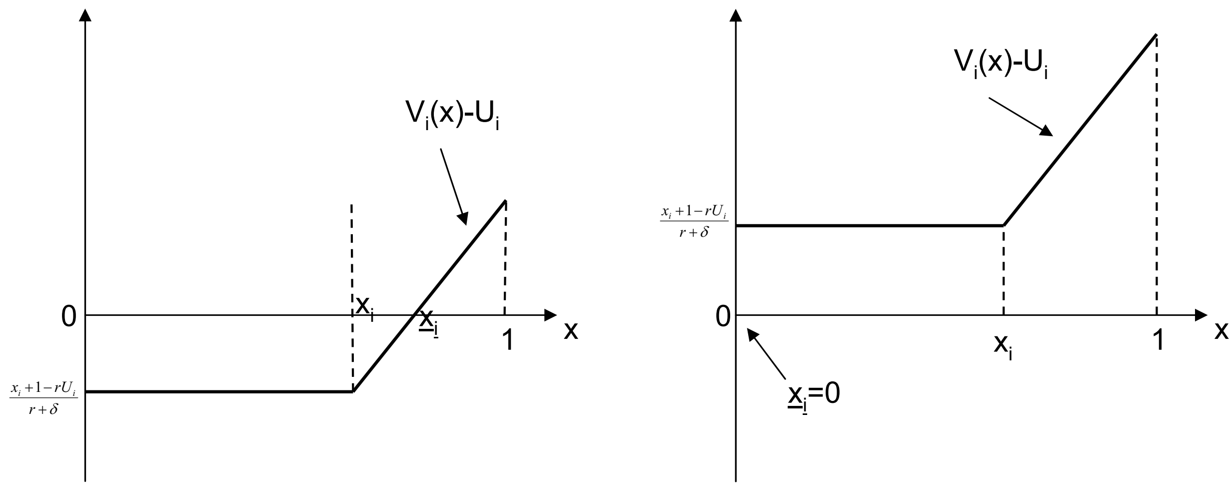

Figure 6: Determination of $\underline{x}_i$

- *$Y2$: $x_i \in [0, \hat{x})$. $\mathcal{M}_i (= \mathcal{F}_i = \mathcal{A}_i)$ is the entire population except for the workers in Group $Y2$.*

X-type can be divided into Group $X1$ and Group $X2$.

- *$X1$: $y_i \in [\hat{y}, 1]$. $\mathcal{M}_i (= \mathcal{F}_i = \mathcal{A}_i)$ is the entire population.*

- *$X2$: $y_i \in [0, \hat{y})$. $\mathcal{M}_i (= \mathcal{F}_i = \mathcal{A}_i)$ is the entire population except for the workers in Group $X2$.*

$\hat{y} = \hat{x}$ from symmetry.

Proof: See Appendix. □

The structure of this equilibrium is depicted in Figure 7. $X1$ and $Y1$ workers can match with anyone they meet. $X2$ and $Y2$ workers match with anyone except the people from their own group. One immediate corollary is that $X2$ and $Y2$ workers always specialize in the tasks that they are good at (as is prescribed in the efficient allocation of Proposition 1). $X1$

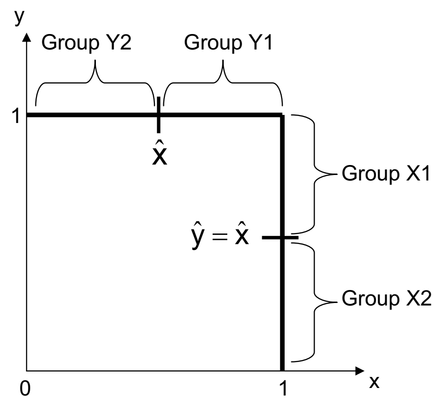

Figure 7: Equilibrium "group" structure

and $Y1$ workers may specialize adversely. The inefficiency due to mismatch takes two forms here. Meeting with the same type (X-type with X-type and Y-type with Y-type) results in unemployment for $X2$ and $Y2$ workers. For $X1$ and $Y1$ workers, mismatch results in an inefficient specialization.

In the context of the general decision rule in Figure 3, $Y1$ workers' decision rule can be depicted as Figure 8 (the entire "flipped-L" is above the reservation value) and $Y2$ workers' decision rule can be depicted as Figure 9. For a $Y2$ worker, the reservation value of x, x_i^R, coincides with $\hat{x}$.

Burdett and Coles (1997), assuming that a person's talent ("pizazz" in their terminology) is one-dimensional, obtained a "class" structure Nash equilibrium.[8] The properties of the equilibrium in our model are strikingly different from the ones of Burdett and Coles' model. First, in our model, a "better" worker is less selective: a Group $X1$ (or $Y1$) worker accepts a match with anyone. In Burdett and Coles (1997), the reservation value for acceptance is an increasing function of a person's talent: therefore, a better worker is more selective.

[8] McNamara and Collins (1990), Eeckhaut (1999), Bloch and Ryder (2000), and Smith (2002) also found the same property.

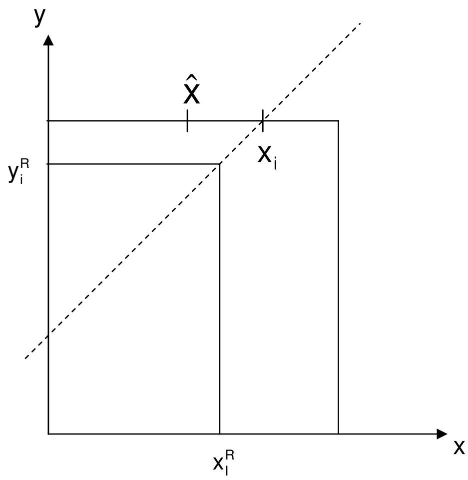

Figure 8: Decision rule of a $Y1$ worker i

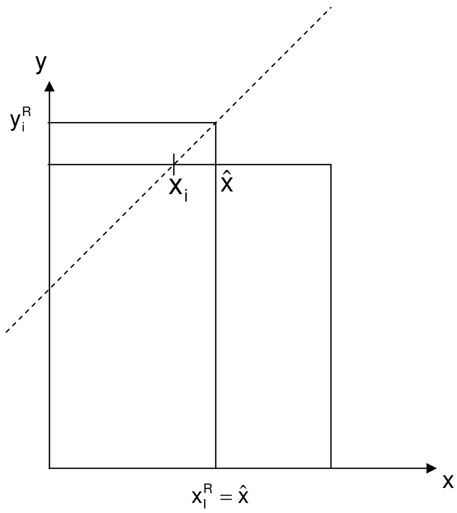

Figure 9: Decision rule of a $Y2$ worker i

This property is discussed more in detail in the next section. Second, there is no one-sided rejection in our model. In Burdett and Coles' model, when a very high-talent person meets a low-talent person, the high-talent person rejects the match while the low-talent person wants to match. This does not occur in our model—when two workers meet, either they both agree to match or both agree not to match.

3.3 Endogenous Talent Distribution of the Unemployed

Now we analyze the stationary equilibrium with endogenous $F(x, y)$. (This is the "market equilibrium" problem in Burdett and Coles [1999] terminology.) Here we assume that the underlying distribution of talent is of "flipped-L" form and uniform. The density at each point is 1/2. The population of Group $Y1$ is $(1-\hat{x})/2$, and the population of Group $Y2$ is $\hat{x}/2$. Similarly, the population of Group $X1$ is $(1-\hat{x})/2$, and the population of Group $X2$ is $\hat{x}/2$.

3.3.1 Stationary Distribution of Employment Status

Given $\hat{x}$, we can calculate the stationary distribution of employment status. Clearly, all the agents in Group 1 ($X1$ and $Y1$) share the same unemployment rate. All the agents in Group 2 ($X2$ and $Y2$) share the same unemployment rate. Denote the employment rate of Group i as m_i and the unemployment rate of Group i as $u_i = 1 - m_i$. The stationarity condition for Group 1 equates the inflow and the outflow of employment. Since a Group 1 worker will match with anyone, the probability of matching is α. The probability of separation is δ by assumption.

$$\alpha(1 - m_1) = \delta m_1.$$

Therefore, $m_1 = \alpha/(\alpha + \delta)$ and $u_1 = \delta/(\alpha + \delta)$.

For Group 2, the probability of matching is lower than α. The total unemployment population is the sum of the Group-1 unemployed and Group-2 unemployed: $u_1(1-\hat{x})/2 + u_2\hat{x}/2$. Among them, they only match with $u_1(1-\hat{x})/2+u_2\hat{x}/4$ amount of people. Therefore, the probability of moving from unemployment to employment for Group-2 workers is $\alpha[u_1(1-$

$\hat{x}) + u_2\hat{x}/2]/[u_1(1-\hat{x}) + u_2\hat{x}]$. The stationarity condition is

$$\alpha\frac{(1-m_1)(1-\hat{x}) + (1-m_2)\hat{x}/2}{(1-m_1)(1-\hat{x}) + (1-m_2)\hat{x}}(1-m_2) = \delta m_2. \tag{6}$$

It turns out that there is only one $m_2 \in [0,1]$ that solves (6).

Proposition 2 *There exists a unique $m_2 \in [0,1]$ that satisfies* (6).

Proof: See Appendix. □

It can also be seen that unskilled workers (Group 2 workers) have a higher unemployment rate than skilled workers (Group 1 workers).[9] This is consistent with the data. (See Mincer [1991].)

Proposition 3 *Group 2 workers have a higher unemployment rate than Group 1 workers, that is, $u_2 > u_1$.*

Proof: See Appendix. □

The solution for m_2 can be expressed as a function of $\hat{x}$. Let us express this dependence by $m_2 = M_2(\hat{x})$. The following holds.

Proposition 4 *m_2 is decreasing in $\hat{x}$, that is, $M_2'(\hat{x}) < 0$.*

Proof: See Appendix. □

3.3.2 Determination of Threshold

Here, we solve for the equilibrium value of $\hat{x}$. From $V_i(\hat{x}) - U_i = 0$,

$$\hat{x} + 1 = rU_i \tag{7}$$

[9] Calling Group 1 workers "skilled" and Group 2 workers "unskilled" under the flipped-L distribution accords with the view that skilled workers are more flexible in performing various tasks. In his classic paper, Schultz (1975) emphasized the importance of people's ability to reallocate their resource in response to changes in economic conditions. In a recent paper, Möbius (2000) built a model where "skill" is defined as the ability to perform wider variety of tasks.

has to be satisfied. From the previous result, we can determine the distribution function $F(x)$. It turns out that for the Y-type,

$$F(x) = \begin{cases} s_1 x & \text{if } x \in [0, \hat{x}) \\ s_2 x + s_1 \hat{x} & \text{if } x \in [\hat{x}, 1), \end{cases}$$

where

$$s_1 = \frac{1}{2} \frac{u_2}{u_1(1-\hat{x}) + u_2 \hat{x}}$$

and

$$s_2 = \frac{1}{2} \frac{u_1}{u_1(1-\hat{x}) + u_2 \hat{x}}. \tag{8}$$

Note that $u_1 = \delta/(\alpha + \delta)$ and $u_2 = 1 - m_2$ can be derived from (6), where $m_1 = 1 - u_1$. Therefore, s_1 and s_2 are functions of $\hat{x}$ and parameters.

From (3), for $x_i = \hat{x}$,

$$rU_i = h + \alpha \left(\int_{\hat{x}}^{1} \frac{1}{r+\delta} [x + 1 - (\hat{x}+1)] s_2 dx + \int_{0}^{1} \frac{1}{r+\delta} [2 - (\hat{x}+1)] \frac{1}{2} dx \right)$$

holds. Calculating this using (7),

$$\hat{x} + 1 = h + \frac{\alpha}{r+\delta} \left[\frac{s_2}{2}(1-\hat{x})^2 + \frac{1}{2}(1-\hat{x}) \right]. \tag{9}$$

This defines $\hat{x}$. The following holds.

Proposition 5 *Under Assumption 1, there exists a unique $\hat{x} \in (0,1)$ that satisfy (9).*

Proof: See Appendix. □

Now, the existence and uniqueness of the market equilibrium are formally shown.

Theorem 2 *The market equilibrium exists, and it is unique.*

Proof: See Appendix. □

3.3.3 Comparative Statics

Here, we conduct comparative statics. First, we show the effect of a change in h (which can be interpreted as the level of unemployment insurance). Second, we construct a numerical example and show the effect of the change in h.

Proposition 6 *$\hat{x}$ is increasing in h.*

Proof: See Appendix. □

From this and Proposition 4, it follows that m_2 is decreasing in h. Total unemployment in the steady-state, $u_1(1-\hat{x})+u_2\hat{x}$, is increasing in h.

3.3.4 Numerical Example

One period corresponds to one month in our model. α is set to 0.33 to match the duration of unemployment of 12 weeks. We set δ to 0.014 to match the employment rate of Group 1 agents to about 4%. We use $r = 0.005$ and $h = 1$. If we solve for (13) and (9) jointly then $\hat{x} = 0.8993$. The employment rate of Group 1 agents (m_1) is 0.959 and the employment rate of Group 2 workers (m_2) is 0.926.

Let $h \in [0.5, 2]$ and compute m_1, m_2, and $\hat{x}$ for different values of h with the baseline parameters. As Figure (10) shows $\hat{x}$ increases as we increase h. Accordingly m_2 decreases. A higher h makes workers more selective in match. Later we will discuss this experiment further in detail.

3.3.5 Income Distribution in the Stationary Equilibrium

Now we look at the distribution of income across workers in the stationary equilibrium discussed above. Since we are focusing on a symmetric equilibrium, we only look at Y-type workers. Clearly, an unemployed (unmatched) worker receives h. In the following, we will discuss the income of employed workers, $w(i,j)$.[10]

[10] Note that there is a variation in income across the workers with the same x_i. This reflects the "luck" factor—some people are luckier than others in finding a good working partner. This variation would appear as "within-group inequality" in the data.

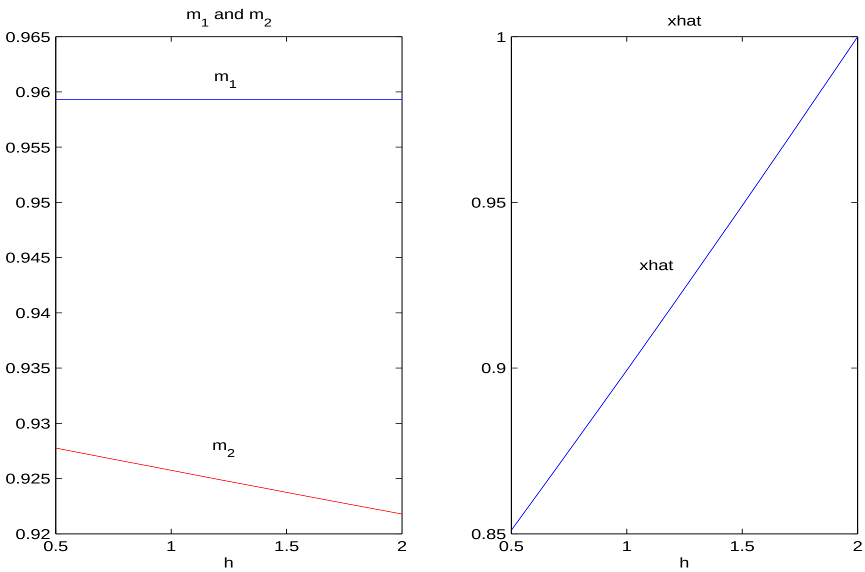

Figure 10: m_1, m_2, and $\hat{x}$ as a function of h.

Fix x_i of the worker i. First we consider the Group-$Y2$ workers, that is, the case of $x_i \in [0, \hat{x})$. Since we are focusing on the stationary equilibrium and the separation rate is constant across the match, the distribution at the new match and the distribution across the population are identical. The distribution at the new match is determined by the relative population of unemployed workers. To construct the distribution function, normalize the total measure of unemployed workers to 1. There are three types of different workers to (potentially) match with from the viewpoint of a Y-type unemployed worker.

1. X-type workers: measure $1/2$
2. Y-type workers with $x_j \in [\hat{x}, 1)$: measure $u_1(1-\hat{x})/2(u_1(1-\hat{x}) + u_2\hat{x})$
3. Y-type workers with $x_j \in [0, \hat{x})$: measure $u_2\hat{x}/2(u_1(1-\hat{x}) + u_2\hat{x})$

With the third group, a match is never formed. Therefore, conditional on the match, the probability of matching with the first group of people is $\Phi_2 = 1/\bar{s}$ where $\bar{s} \equiv 1 + u_1(1-\hat{x})/(u_1(1-\hat{x}) + u_2\hat{x})$. The conditional distribution of matching with the second group worker with $x_j = x$ is uniform with density $\phi_1^2 = u_1/\bar{s}(u_1(1-\hat{x}) + u_2\hat{x})$. Therefore, for the workers with $x_i \in [0, \hat{x})$, the income distribution (conditional on employment) follows

1. Density ϕ_1^2 with $w(i,j) = 1 + x_j$ for $x_j \in [\hat{x}, 1)$,

2. A mass of Φ_2 with $w(i,j) = 2$.

Second, we consider Group-$Y1$ workers. They face the same three types of potential partners as before. One difference from above is that they will form a match with anyone. Another difference is that when matched with a partner j in the second group above, i may specialize in task X or task Y depending on whether $x_i \geq x_j$.

The conditional probability of matching with the first group of people is $\Phi_1 = 1/2$. The conditional density of matching with the second group worker is $\phi_1^1 = u_1/2(u_1(1-\hat{x}) + u_2\hat{x})$ and the conditional density of matching with the third group worker is $\phi_2^1 = u_2/2(u_1(1-\hat{x}) + u_2\hat{x})$. Therefore, for workers with $x_i \in [\hat{x}, 1)$, the income distribution (conditional on employment) follows:

1. A mass of $\phi_2^1\hat{x} + \phi_1^1(x_i - \hat{x})$ with $w(i,j) = 1 + x_i$,

2. Density ϕ_1^1 with $1 + x_j$ with $w(i,j) = 1 + x_j$ for $x_j \in (x_i, 1)$,

3. A mass of Φ_1 with $w(i,j) = 2$.

Figure 11 depicts the cumulative income distributions of the workers (including the unemployed workers, who receive h) given x_i, in the benchmark case. This is the distribution by the pure "luck" factor. The dashed line is for $x_i = 0.95$, who are the Group-1 workers. The solid line is for $x_i = 0.7$, who are the Group-2 workers. (In fact, all the Group-2 workers have the identical distribution functions.) We can see that there are more workers with $w(i,j) = 1$ for $x_i = 0.7$, since they are more selective. There are more "medium-range" income earners for $x_i = 0.95$.

Figure 12 draws the cumulative income distribution for $x_i = 0.7$ (Group 2) when h changes.[11] There are two differences: There are more unemployment with $h = 1.4$, since

[11] Note that here we are ignoring how h is financed, to make the comparison clear.

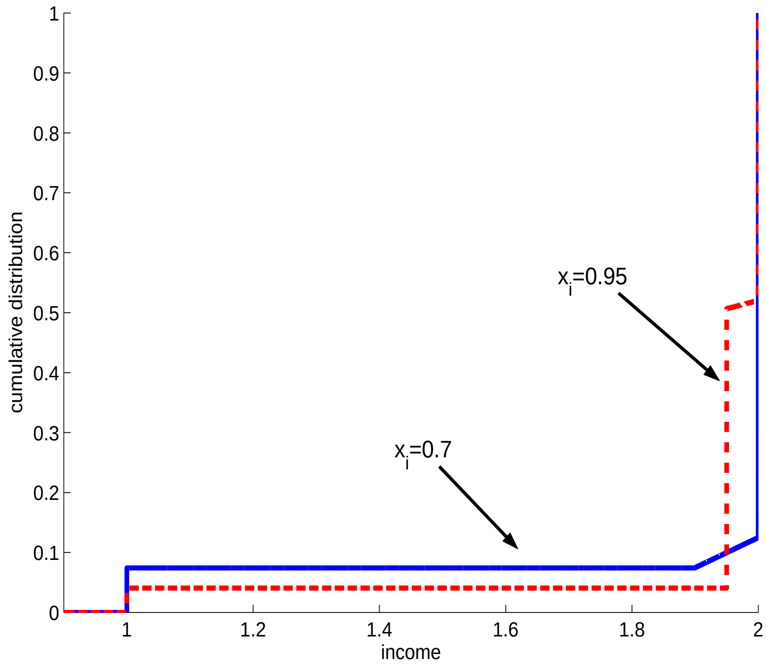

Figure 11: Cumulative distribution functions of income when $h = 1.0$

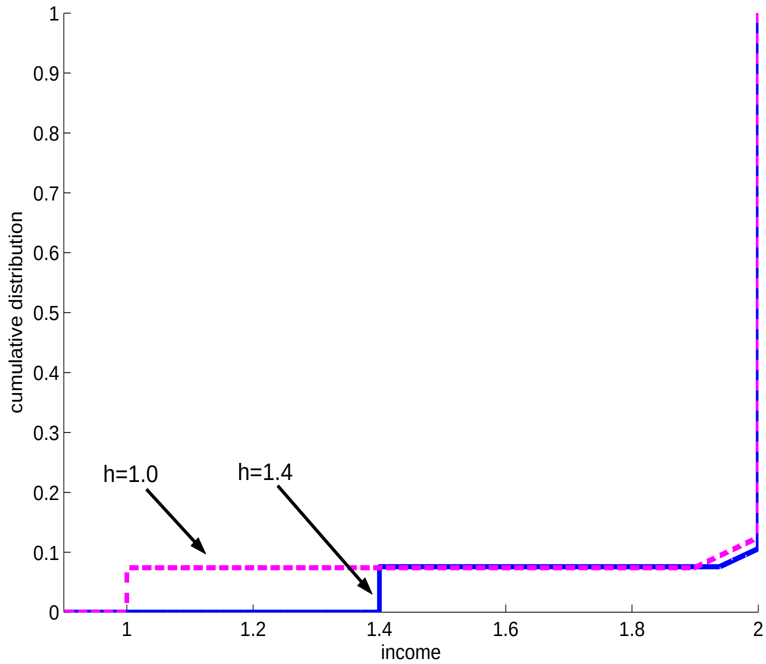

Figure 12: Cumulative distribution functions of income with $x_i = 0.7$

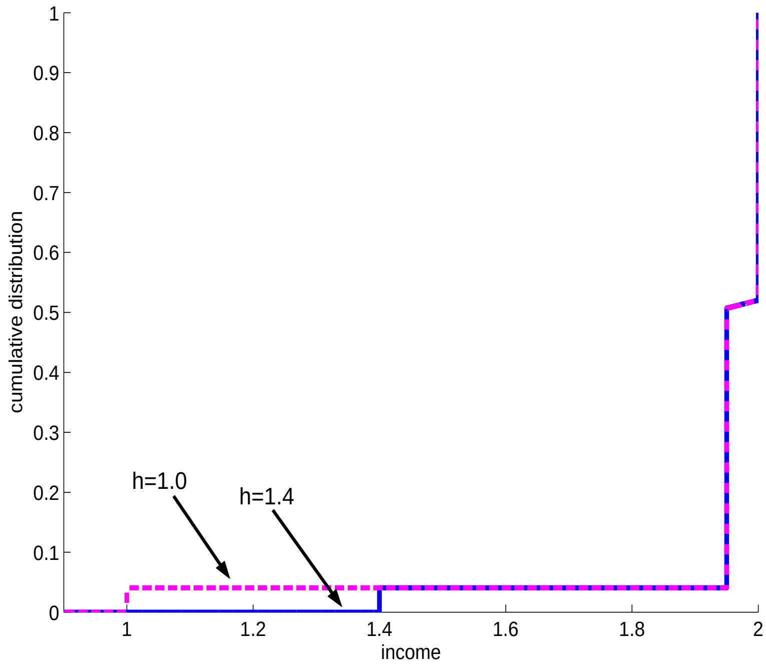

Figure 13: Cumulative distribution functions of income with $x_i = 0.95$

people are more selective. By the same effect, employed workers earn more with $h = 1.4$. In particular, we can show that from (6), $\Phi_1 = \alpha u_2/[2\delta(1-u_2)]$ and therefore increasing in u_2 (thus increasing in h). As h increases, there are more Group-2 workers earning the income 2. On average, these two effects provide a mixed effect on the income of the Group-2 workers.

Figure 13 draws the same graphs as Figure 12 for $x_i = 0.95$. The two distributions are very similar. The graph of $h = 1.4$ lies above for the income level between $1+x_i$ and 2. (That is, the distribution for $h = 1.4$ is dominated in this region.) This follows since ϕ_1^1 is decreasing in h (u_1 is constant and $u_1(1-\hat{x}) + u_2\hat{x}$ is increasing in h). Intuitively, when h increases, both u_2 and $\hat{x}$ increases. Thus there are more Group-2 workers in the unemployment pool. This worsens the matching perspective of a Group-1 worker. In sum, high h tends to lower the income for Group-1 workers.

Next, we turn to the difference of income *across* different x_i. Figure 14 depicts the average income across different x_i in the benchmark case. The average income is calculated

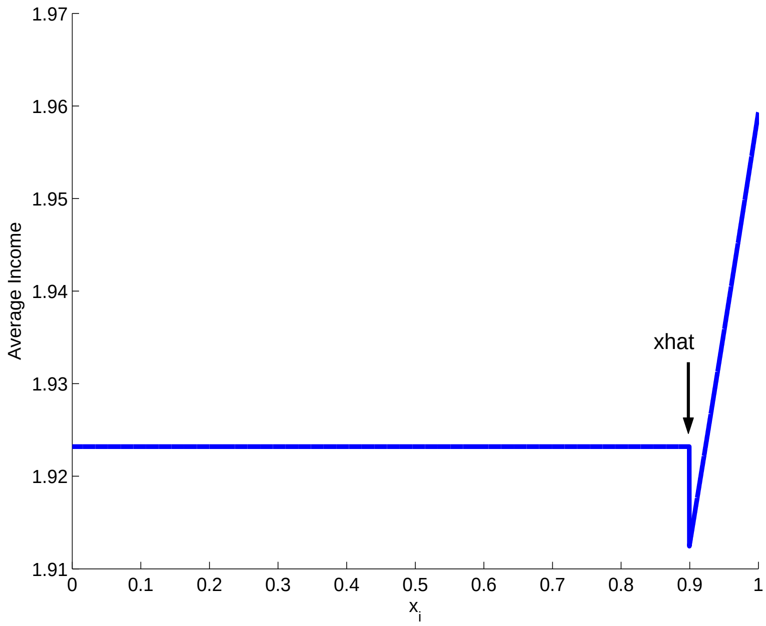

Figure 14: Average income across different x_i

as $[\int_{\hat{x}}^{1}(1+x')\phi_1^2 dx' + 2\Phi_2]m_2 + hu_2$ for $x_i < \hat{x}$ and $[(1+x_i)[\phi_2^1\hat{x} + \phi_1^1(x_i - \hat{x})] + \int_{x_i}^{1}(1 + x')\phi_1^1 dx' + 2\Phi_1]m_1 + hu_1$ for $x_i \geq \hat{x}$.

Note that there is a "drop" in the average income at $x_i = \hat{x}$. This is due to the fact that we are comparing the steady-state. At $x_i = \hat{x}$, the workers are indifferent between being selective or not. If all the non-selective workers decide to be selective, many of them (workers who are matched with $x_j < \hat{x}$) have to suffer from unemployment initially. They earn a higher average income in the long run, and this makes them indifferent. A worker with $x_i = \hat{x} - \epsilon$ earns higher average income in the long run than a worker with $x_i = \hat{x} + \epsilon$, simply because he is more selective.

Figure 15 draws the same relationship for different h. Clearly, higher h means that average income is higher overall, since the income during unemployment is higher and people are more selective. However, again, this figure ignores that it is costly to finance h. Figure 16 makes an adjustment to make the comparison "fair"—there, $h(u_1(1-\hat{x}) + u_2\hat{x})$ is subtracted from each

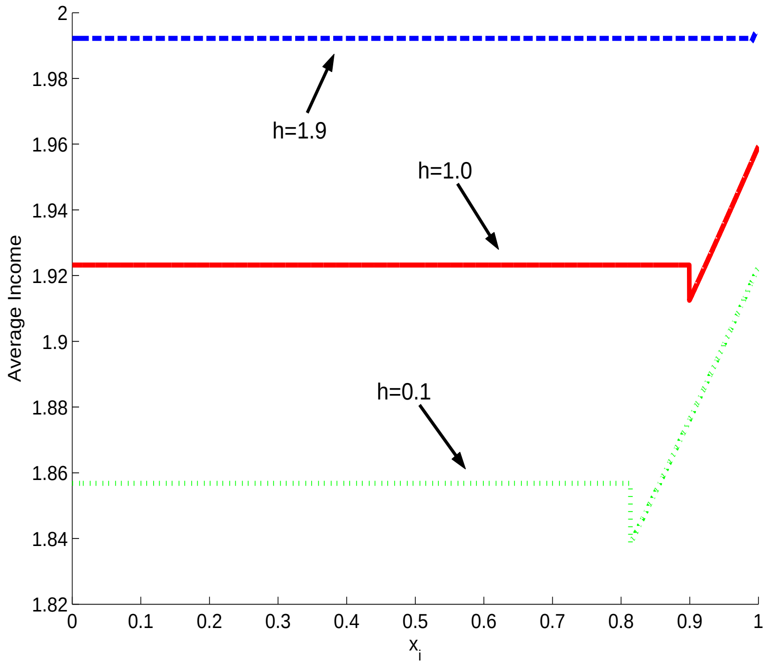

Figure 15: Average income across different x_i

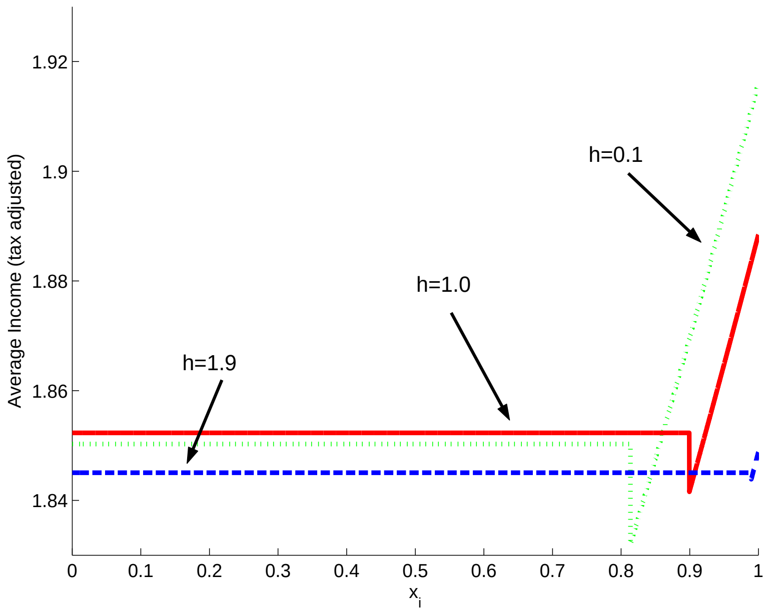

Figure 16: Average income across different x_i, tax adjusted

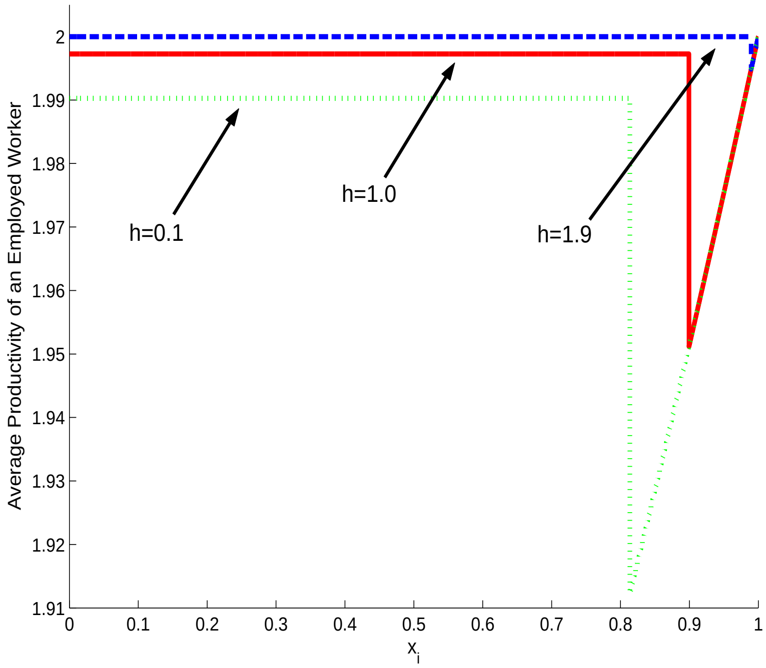

Figure 17: Average production across different x_i, tax adjusted

worker's income so that h is financed by a balanced budget. Interestingly, changing h has a non-monotonic effect on the average income, even after adjusting for the tax. This is due to the fact that higher h makes people more selective, which raises the average productivity of Group-2 employed workers. In sum, a worker with high x_i is always against the high h. However, a worker with low x_i may prefer to have high h.

To see this point more clearly, Figure 17 draws a similar diagram for the average production, that is, the average income given that one is employed. At the right tail, the average production is slightly decreasing in h, but in other parts they are significantly increasing in h.

3.3.6 Aggregate Output

The aggregate output can be calculated by summing up all the production by the individuals. In particular, the aggregate output Q_A is:

$$Q_A = 2 \times (Q_{Y1} + Q_{Y2}),$$

where Q_{Y1} is the aggregate income of Group $Y1$ and Q_{Y2} is the aggregate income of Group $Y2$.

Q_{Y2} can be calculated as:

$$\begin{aligned} Q_{Y2} &= \int_0^{\hat{x}} \left(\int_{\hat{x}}^1 (1+x')\phi_1^2 dx' + 2\Phi_2 \right) \frac{m_2}{2} dx \\ &= \frac{m_2 \hat{x}}{2} \left(\phi_1^2 \left[\frac{3}{2} - \hat{x} - \frac{\hat{x}^2}{2} \right] + \frac{2}{s} \right). \end{aligned}$$

Q_{Y1} can be calculated as:

$$\begin{aligned} Q_{Y1} &= \int_{\hat{x}}^1 \left((1+x)[\phi_2^1 \hat{x} + \phi_1^1 (x - \hat{x})] + \int_x^1 (1+x')\phi_1^1 dx' + 2\Phi_1 \right) \frac{m_1}{2} dx \\ &= \frac{m_1}{2} \left(\phi_2^1 \left[\frac{3}{2}\hat{x} - \hat{x}^2 - \frac{\hat{x}^3}{2} \right] + \phi_1^1 \left[\frac{10}{6} - 3\hat{x} + \hat{x}^2 + \frac{\hat{x}^3}{3} \right] + 1 - \hat{x} \right). \end{aligned}$$

Figure 18 calculates Q_A in the above numerical experiment. It can be seen that when the level of h is low, an increase in h (which can be viewed as a more generous unemployment insurance policy) can increase the total output, despite the fact that the unemployment rate always increases with h. Q_A may increase since when h increases, $\hat{x}$ increases, and a more efficient pattern of specialization realizes. In fact, the average productivity of workers $Z \equiv Q_A/(m_1[1-\hat{x}] + m_2\hat{x})$ increases as h increases. See Figure 19. As h approaches to 2, Z also approaches to 2. When h is (very close to) 2, $\hat{x}$ becomes 1 (see Figure 10) and everyone becomes a Group-1 worker. Workers match only with the opposite-type workers. When h exceeds 2, no one want to match and the aggregate output collapses to zero.

Z is monotonically increasing in h, but when h is too high, this productivity gain is overwhelmed by the increase in unemployment, and Q_A starts to fall after some value of h. This example shows that through the channel of specialization, the unemployment in-

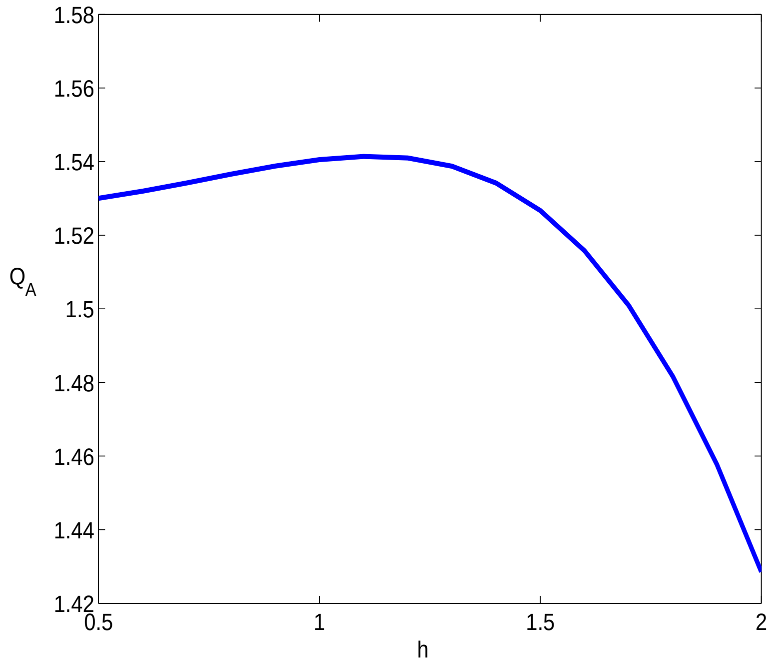

Figure 18: Q_A as a function of h.

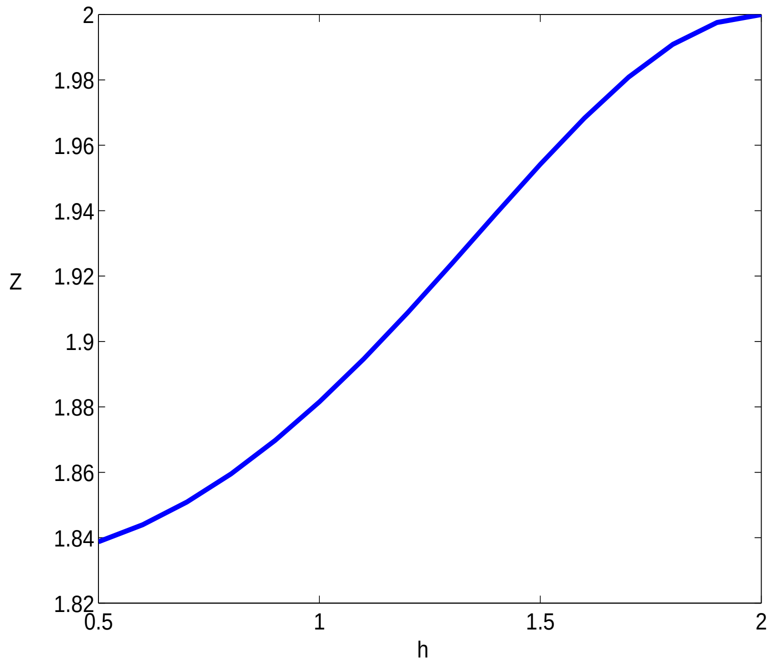

Figure 19: Z as a function of h.

surance policy can affect the aggregate productivity and output. Several papers[12] have also found that the *average productivity of workers* may increase by more generous unemployment insurance. However, we are aware of only one other paper (Acemoglu and Shimer [1999]) that constructed a model where *aggregate output* increases by more generous unemployment insurance. Aggregate output is the product of the average productivity and the number of workers, and in most of the models the increase of the average productivity is not large enough to compensate the decrease of the number of the workers. Acemoglu and Shimer's (1999) mechanism is based on the risk aversion of the workers, and very different from ours. In our model, the productivity gain comes from a better allocation of talent.

4 Discussion: Are Better Workers Always Less Selective?

The example above has a virtue of the simplicity of the solution. However, the distribution is special and certain results can be seen as extreme. In this section, we will discuss one such a result in detail.

In the previous section, it was shown that a better worker is less selective in matching. This result does not always hold for other distributions of talents. Consider the following two-types example. Assume that in the economy, there are two types, G and B. G-type's talent vector is $(1, 1)$ and B-type's talent vector is (k, k), where $k \in (0, 1)$. Note that in this framework, specialization is not important, since both workers have the same skill between two tasks.

Let π_G be the fraction of G-type workers in the unemployment pool. $(1 - \pi_G)$ is the

[12] Diamond (1981) suggests a mechanism related to ours through various setup costs for production. As the unemployment insurance becomes more generous, workers become more selective and choose matches with lower setup costs. His model has ex-ante homogeneous workers. Marimon and Zilibotti (1999) proposes a similar mechanism: workers become more selective when the unemployment insurance is more generous. They consider a model with symmetric workers (with their characteristics lying on a circle), and in the equilibrium they consider, the unemployment rate is uniform across types. Acemoglu (2001) focuses on the "hold-up" problem in search equilibrium and argues that unemployment insurance may increase the fraction of high-paying jobs with large sunk costs. Lagos (2004) constructs a model where policies can affect the productivity through search frictions. In his paper, it is also the case that a more generous unemployment insurance increases average productivity. The mechanism in his paper is that when the unemployment insurance increases, an inefficient match becomes unlikely to survive.

fraction of B-type workers in the unemployment pool. Since we are only interested in the matching decisions here, let us assume that π_G is given exogenously.[13]

Type G will select from two possible decisions:

A Accept all,

B Accept only G.

The Bellman equations for a G-type worker is:

$$rV_G^G = 2 - \delta(V_G^G - U_G),$$

$$rV_G^B = 1 + k - \delta(V_G^B - U_G),$$

$$rU_G = h + \alpha[\pi_G(V_G^G - U_G) + (1 - \pi_G)(V_G^B - U_G)],$$

where V_i^j is the value of a i-type worker matched with a j-type worker and U_i is the value of an unemployed Type i worker.

G-type will choose **A** if and only if $V_G^B - U_G \geq 0$. From the Bellman equations, it can be calculated that $V_G^B - U_G \geq 0$ holds if and only if $h \leq R_G$, where

$$R_G = 1 + k - \frac{\alpha}{r + \delta}(1 - k)\pi_G.$$

Similarly, B-type has two possible decisions:

a Accept all,

b Accept only G.

Type B may or may not be accepted by G. Therefore, this type's decision-making process will have to be conditional on the behavior of G.

When G-type chooses **A**, B-type chooses **a** if and only if $h \leq R_B$, where

$$R_B = 2k - \frac{\alpha}{r + \delta}(1 - k)\pi_G.$$

[13]This can be achieved by replacing the matched pair by their (unmatched) "clones" every time a match is formed.

Note that $R_G > R_B$. When G-type chooses $\mathbf{B}$, B-type has no choice other than $\mathbf{a}$ (as long as $2k > h$).

Thus, there are three types of equilibrium, depending on the value of h. When h is lower than R_B, the equilibrium set of choices is $(\mathbf{A}, \mathbf{a})$. Both are not selective. When $R_B < h < R_G$, the equilibrium set of choices is $(\mathbf{A}, \mathbf{b})$. This situation is similar to the example in the previous section—better workers are less selective. When $h > R_G$, the equilibrium set of choices is $(\mathbf{B}, \mathbf{a})$. In this case, better workers are more selective—similar to the result of Burdett and Coles (1997).

As can be seen from this example, in general a better worker can be more selective or less selective in our framework. (Note that in Burdett and Coles (1997), a better worker is *always* more selective.) The example in the previous section can be viewed as the opposite extreme of Burdett and Coles (1997)-type result.

5 Conclusion

This paper constructed a model of matching with heterogeneous workers. Unemployed workers randomly meet with each other, and decide whether or not to accept the match. After matching, one specializes in one task, and the other specializes in the other task, following their comparative advantage. Under the matching friction, the resulting pattern of specialization in the economy is not necessarily efficient.

When the underlying talent distribution is a "flipped-L" shape, the Nash equilibrium exhibits an endogenous "group" structure—some workers accept a match with anyone, and some workers accept a match with anyone except for those from their own group. It was shown that a more generous unemployment insurance system may increase aggregate output. This is due to a better allocation of talent. Unemployment insurance induces people to specialize in what they are good at.

Appendix

A Equivalence to the Model with Tax

Suppose that the unemployment insurance is financed by a lump-sum tax τ on all workers. Then, (1) and (2) become:

$$rU_i = h - \tau + \alpha \int_{\mathcal{F}_i} \max\langle V_i(x_j, y_j) - U_i, 0\rangle dF(x_j, y_j) \tag{10}$$

and

$$rV_i(x_j, y_j) = \max\langle x_i + y_j, x_j + y_i\rangle - \tau - \delta[V_i(x_j, y_j) - U_i]. \tag{11}$$

Let us define $\tilde{U}_i \equiv U_i + \tau/r$ and $\tilde{V}_i(x_j, y_j) \equiv V_i(x_j, y_j) + \tau/r$. Then, (10) and (11) can be rewritten as

$$r\tilde{U}_i = h + \alpha \int_{\mathcal{F}_i} \max\langle \tilde{V}_i(x_j, y_j) - \tilde{U}_i, 0\rangle dF(x_j, y_j)$$

and

$$r\tilde{V}_i(x_j, y_j) = \max\langle x_i + y_j, x_j + y_i\rangle - \delta[\tilde{V}_i(x_j, y_j) - \tilde{U}_i],$$

which are equivalent to (1) and (2).

If we assume that the government has to balance the budget every period, $\tau = hu$ has to hold. The equilibrium value of τ is determined by the equilibrium value of u.

Note that if the tax is levied only on the employed workers, this equivalence does not hold. Moreover, this type of tax can become a source of multiple equilibria. When the workers expect that the tax will be high, the workers become more selective and the unemployment rate becomes high. High unemployment rate means that the tax is high in equilibrium, and the expectation is self-fulfilled.

B Proofs of Theorems and Propositions in Section 3

Proof of Proposition 1:

Suppose, by contradiction, there are matches between i with $x_i > y_i$ and j with $x_j > y_j$. Since the distribution of match is symmetric, this implies that there are matches between k

with $x_k < y_k$ and l with $x_l < y_l$. Without loss of generality, assume that i and k specialize in x and j and l specialize in y. Then the total output of these two matches is $x_i + y_j + x_k + y_l$. Now, instead match i with k and j with l, and let i and j specialize in x and k and l specialize in y. Then the total output becomes $x_i + y_k + x_j + y_l$. Since $x_j > y_j$ and $y_k > x_k$, this is strictly larger than before. Contradiction. $\square$

Proof of Theorem 1:

We prove the theorem for Y-type workers. The argument for X-type workers are symmetric.

Existence:

Step 1: Find a candidate for $\hat{x}$

Pick a value of $\bar{x} \in (0, 1)$. Assume the following behavior for everyone (denote the worker j) except for worker i.

- If $x_j \in [\bar{x}, 1]$, $\mathcal{A}_j = [0, 1]$.
- If $x_j \in [0, \bar{x})$, $\mathcal{A}_j = [\bar{x}, 1]$.

If we can find an $\bar{x}$ that makes the worker i to behave in the same way, we call it $\hat{x}$ and we are done.

So, let's look at i's behavior. Note that since the other workers behave as above, the feasible set for i is as following.

- If $x_i \in [\bar{x}, 1]$, $\mathcal{F}_i = [0, 1]$.
- If $x_i \in [0, \bar{x})$, $\mathcal{F}_i = [\bar{x}, 1]$.

First, given $\bar{x}$, we look for a worker whose $\underline{x}_i = x_i$. Call this value $\tilde{x}_i$. Does $\tilde{x}_i$ always exist? For such a worker, $V_i(x) - U_i$ looks as in Figure 20. Therefore, from (5), $rU_i = \tilde{x}_i + 1$ has to hold. From (3), $\tilde{x}_i$ is a solution to

$$x_i + 1 = h + \frac{\alpha}{r + \delta} \int_{\mathcal{M}_i} (x_j - x_i) dF(x_j). \tag{12}$$

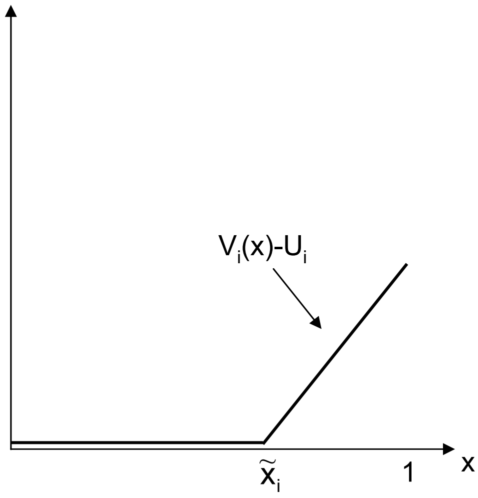

Figure 20: Determination of $\tilde{x}_i$

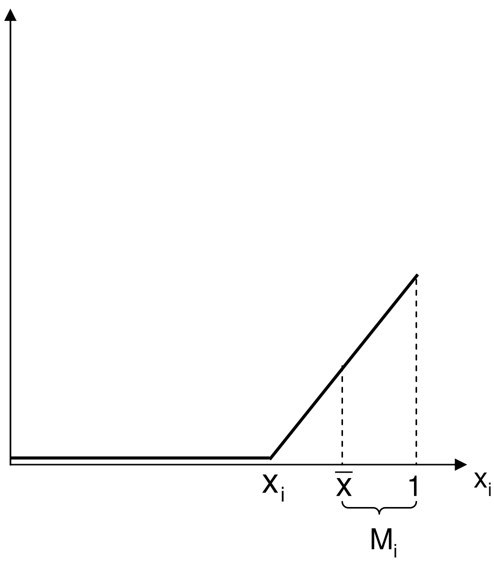

Figure 21: The range of $\mathcal{M}_i$ for $x_i \leq \bar{x}$

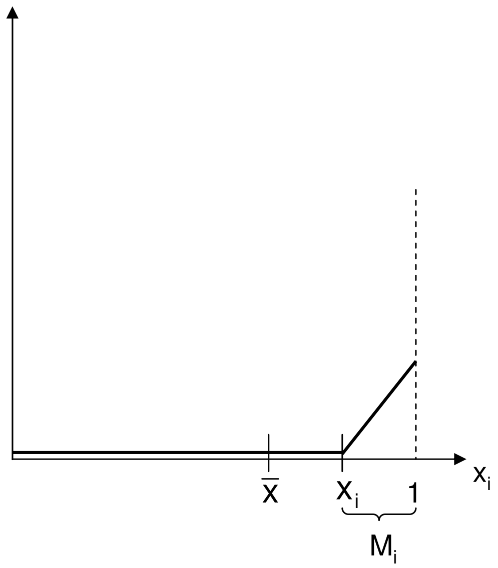

Figure 22: The range of $\mathcal{M}_i$ for $x_i > \bar{x}$

The left-hand-side of (12) is a linear and continuous function of x_i. It can easily be seen that the right-hand-side is a monotonically decreasing and continuous function of x_i. (Note that $\mathcal{M}_i$ is a function of x_i. Specifically, $\mathcal{M}_i = \mathcal{F}_i \cap [x_i, 1] = [\max\{\bar{x}, x_i\}, 1]$. See Figures 21 and 22.) The right-hand-side takes the value of $g(\bar{x}) \equiv h + \alpha \int_{[\bar{x},1]} x_j dF(x_j)/(r+\delta)$ when $x_i = 0$. It takes the value of h when $x_i = 1$: this can be seen by moving $\tilde{x}_i$ to 1 in Figure 20 — inside the integral will converge to zero for all x_j. $g(\bar{x})$ is monotonically decreasing in $\bar{x}$ and approaches to $h + \alpha/2(r+\delta)$ as $\bar{x}$ approaches to 1 from below (note that there is a mass of 1/2 at $x_j = 1$). Therefore, when Assumption 1 is satisfied, there always exists a unique $\tilde{x}_i \in (0, 1)$ that satisfies (12) (See Figure 23).

Moreover, as we change $\bar{x}$, the right-hand-side of (12) shifts down continuously. Therefore, $\tilde{x}_i$ is a continuous and decreasing function of $\bar{x}$. From the Brouwer Fixed-Point Theorem (Stokey and Lucas with Prescott [1989] Theorem 17.3), there exists a $\bar{x}$ (in fact, this is the unique fixed point) that satisfies $\bar{x} = \tilde{x}_i$ (See Figure 24). We claim that this fixed point is the $\hat{x}$ that we are looking for.

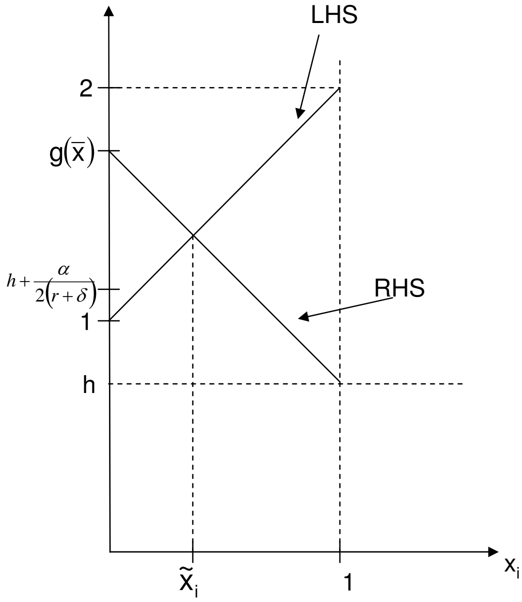

Figure 23: Determination of $\tilde{x}_i$

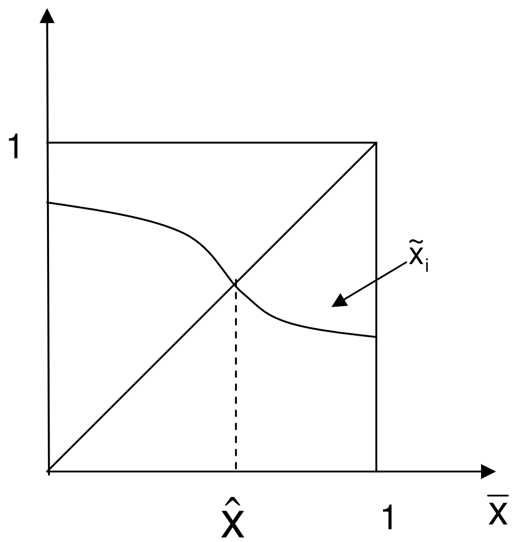

Figure 24: Determination of $\hat{x}$

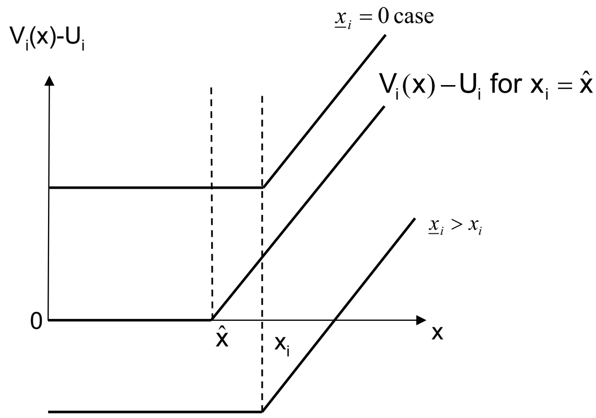

Figure 25: Case 1

Step 2: Verify the Nash equilibrium

To see this, consider i's behavior when $\bar{x}$ is given as this value (we call this $\hat{x}$ hereafter). Note that when $x_i = \hat{x}$, the worker is indifferent among any $\underline{x}_i \in [0, \hat{x}]$.

Case 1 $x_i > \hat{x}$:

To show that $\underline{x}_i = 0$, it suffices to prove a contradiction for $\underline{x}_i \geq x_i$. (From Figure 25, it can be seen that $\underline{x}_i$ cannot belong to $(0, x_i)$ since $V_i(x) - U_i$ cannot cross zero at a point belonging to $(0, x_i)$.) So, suppose that $\underline{x}_i \geq x_i$ (the lower graph in Figure 25.) Then, the right-hand-side of (3) is smaller for this x_i compared to the case when $x_i = \hat{x}$ (the middle graph in Figure 25). This contradicts to the fact that U_i is increasing in x_i.

Case 2 $x_i < \hat{x}$:

To show that $\underline{x}_i = \hat{x}$, we have to rule out the three cases: $\underline{x}_i = 0$, $\underline{x}_i \in [x_i, \hat{x})$, and $\underline{x}_i > \hat{x}$. (From Figure 26, it can be seen that $\underline{x}_i$ cannot belong to $(0, x_i)$ since $V_i(x) - U_i$ cannot cross zero at a point belonging to $(0, x_i)$.) Suppose that $\underline{x}_i = 0$. Then, the right-hand-side of (3) is larger for this x_i compared to the case when $x_i = \hat{x}$. This contradicts to the fact that U_i is increasing in x_i. Next suppose that $\underline{x}_i \in [x_i, \hat{x})$.

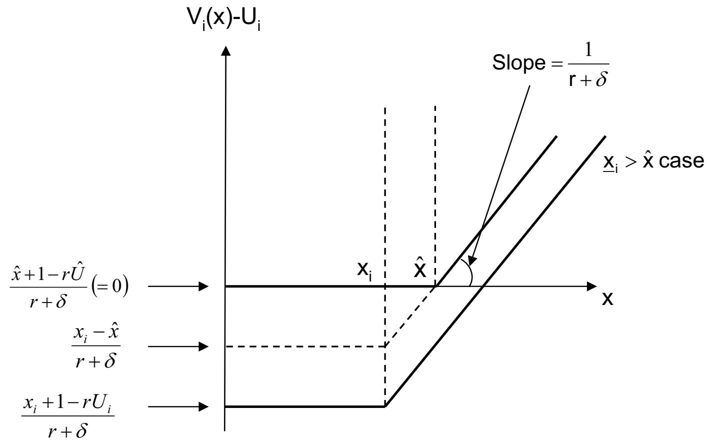

Figure 26: Case 2

Again, the right-hand-side of (3) is larger for this x_i compared to the case when $x_i = \hat{x}$. This contradicts to the fact that U_i is increasing in x_i. Finally, suppose $\underline{x}_i > \hat{x}$ (the lower graph in Figure 26). Call the value of U_i when $x_i = \hat{x}$ as $\hat{U}$. Then, $\underline{x}_i > \hat{x}$ implies

$$\frac{x_i + 1 - rU_i}{r + \delta} < \frac{x_i - \hat{x}}{r + \delta}.$$

(See the vertical axis of Figure 26.) Therefore, $(\hat{x} + 1 - rU_i)/(r + \delta) < 0$. Since $(\hat{x} + 1 - r\hat{U})/(r + \delta) = 0$, this implies $U_i > \hat{U}$. This contradicts to the fact that U_i is increasing in x_i.

We've shown the existence.

Uniqueness:

First, given an arbitrary $\mathcal{F}(x_i)$, consider the determination of $\mathcal{A}(x_i)$. Note that from Figure 6, if there exists an x_i and $x_j \leq x_i$ such that $x_j \in \mathcal{A}(x_i)$, $\mathcal{A}(x_i) = [0, 1]$. (Let $\hat{x}$ be the smallest x_i that this happens.)

This in turn implies that $\mathcal{F}(x_i)$ is identical above the 45-degree line. Note that for $x_i < \hat{x}$, since $\mathcal{A}(x_i)$ does not extend below x_i, $\mathcal{M}(x_i) \subseteq \mathcal{F}(x_i)$. In fact, using a similar logic as Case 2 above, it is straightforward to prove that $\mathcal{A}(x_i) = [\hat{x}, 1]$ for $x_i < \hat{x}$. (First, use $(\hat{x}+1-r\hat{U})/(r+\delta) \geq 0$ instead of $(\hat{x}+1-r\hat{U})/(r+\delta) = 0$ in Case 2 to prove that $\mathcal{A}(x_i) \supseteq [\hat{x}, 1]$. Since $\mathcal{A}(x_i)$ does not extend below x_i for $x_i < \hat{x}$ that is arbitrarily close to $\hat{x}$ and $\mathcal{A}(x_i)$ is identical for $x_i < \hat{x}$, $\mathcal{A}(x_i) = [\hat{x}, 1]$.)

This implies that $\mathcal{F}(x_i) = [0, 1]$ for $x_i \geq x_i$. Given this, from the same logic as Case 1 above, it is straightforward to prove that $\mathcal{A}(x_i) = [0, 1]$ for $x_i \geq \hat{x}$. (Therefore, $\mathcal{M}(x_i) = [0, 1]$ for $x_i \geq x_i$.)

This implies that $\mathcal{F}(x_i) = [\hat{x}, 1]$ for $x_i < \hat{x}$. Since $\mathcal{A}(x_i) = [\hat{x}, 1]$ for $x_i < \hat{x}$, $\mathcal{M}(x_i) = [\hat{x}, 1]$ for $x_i < \hat{x}$.

Above we established that the equilibrium, if exists, has to exhibit the "group" structure described in the Theorem. Since the uniqueness of $\hat{x}$ is shown in the existence proof, the Nash equilibrium is unique. $\square$

Proof of Proposition 2:

Using the solution for m_1, (6) can be reduced to a quadratic equation $f(m_2) = 0$, where

$$f(m_2) \equiv \left(\frac{\alpha}{2} + \delta\right) \hat{x} m_2^2 - (\alpha\hat{x} + \delta)m_2 + \frac{\alpha}{2}\hat{x} + \frac{\alpha\delta}{\alpha+\delta}(1-\hat{x}). \tag{13}$$

Since $f(0) = \alpha\hat{x}/2 + \alpha\delta(1-\hat{x})/(\alpha+\delta) > 0$ and $f(1) = -\delta^2(1-\hat{x})/(\alpha+\delta) < 0$, there is a unique $m_2 \in [0, 1]$ that satisfy $f(m_2) = 0$. $\square$

Proof of Proposition 3:

We will show that $m_2 < m_1$ by exhibiting $f(m_1) < 0$ in (13). Using the fact that $m_1 = \alpha/(\alpha+\delta)$, $f(m_1)$ can be reduced to $f(m_1) = -\alpha\delta^2\hat{x}/2(\alpha+\delta)^2 < 0$. $\square$

Proof of Proposition 4:

Let

$$G(m_2, \hat{x}) \equiv \left(\frac{\alpha}{2} + \delta\right) \hat{x} m_2^2 - (\alpha\hat{x} + \delta)m_2 + \frac{\alpha}{2}\hat{x} + \frac{\alpha\delta}{\alpha+\delta}(1-\hat{x}).$$

Then,

$$\begin{aligned}
\partial G/\partial m_2 &= 2\left(\alpha/2+\delta\right)\hat{x}m_2-\left(\alpha\hat{x}+\delta\right) \\
&= (\alpha+\delta)\hat{x}m_2+\delta\hat{x}m_2-\alpha\hat{x}-\delta \\
&< (\alpha+\delta)\hat{x}m_1+\delta\hat{x}m_2-\alpha\hat{x}-\delta \\
&= \alpha\hat{x}+\delta\hat{x}m_2-\alpha\hat{x}-\delta \\
&< 0
\end{aligned}$$

and

$$\begin{aligned}
\partial G/\partial \hat{x} &= \left(\alpha/2+\delta\right)m_2^2-\alpha m_2+\alpha/2-\alpha\delta/(\alpha+\delta) \\
&= [(\alpha\hat{x}+\delta)m_2-\alpha\hat{x}/2-\alpha\delta(1-\hat{x})/(\alpha+\delta)]/\hat{x}-\alpha m_2+\alpha/2-\alpha\delta/(\alpha+\delta) \\
&= \delta m_2/\hat{x}-\alpha\delta(1-\hat{x})/(\alpha+\delta)\hat{x}-\alpha\delta/(\alpha+\delta) \\
&= [(\alpha+\delta)\delta m_2-\alpha\delta(1-\hat{x})-\alpha\delta\hat{x}]/(\alpha+\delta)\hat{x} \\
&= [(\alpha+\delta)\delta m_2-\alpha\delta]/(\alpha+\delta)\hat{x} \\
&< [(\alpha+\delta)\delta m_1-\alpha\delta]/(\alpha+\delta)\hat{x} \\
&= 0.
\end{aligned}$$

From the Implicit Function Theorem, $M_2'(\hat{x})<0$. □

Proof of Proposition 5:

Define

$$H(\hat{x}) \equiv h+\frac{\alpha}{r+\delta}\left[\frac{s_2}{2}(1-\hat{x})^2+\frac{1}{2}(1-\hat{x})\right]-\hat{x}-1.$$

The solution to the quadratic equation $H(\hat{x})=0$ is the solution to (9). Under Assumption 1, $H(0)>0$ and $H(1)$¡1. Therefore, there exists a unique $\hat{x}\in(0,1)$ that satisfy (9). □

Proof of Theorem 2:

(Again, we look at only type Y workers.) Since the underlying talent distribution function is continuous, the stationary talent distribution of the unemployed is also continuous (no mass point). In Theorem 1, we established that for given distribution of the unemployed, there exists a unique Nash equilibrium that can be characterized by $\hat{x}$.

Given this Nash equilibrium, the distribution of unemployed is characterized by m_2, which is characterized by (6). Therefore, (6) determines the mapping from $\hat{x}$ to m_2. Proposition 2 ensures that m_2 exists (and unique for given $\hat{x}$), and Proposition 4 ensures that m_2 is decreasing in $\hat{x}$. From (6), it is clear that m_2 is continuous in $\hat{x}$.

Given m_2 (and therefore s_2), (9) determines the value of $\hat{x}$ in Nash equilibrium. Proposition 5 guarantees the existence and uniqueness of $\hat{x}$ given m_2. By some algebra, it is possible to show that $\hat{x}$ is increasing in m_2. From (8) and (9), it is clear that $\hat{x}$ is continuous in m_2.

Therefore, the combination $(\hat{x}, m_2)$ that satisfies (6) and (9) always exists, and is unique. This establishes the existence and uniqueness of the market equilibrium. □

Proof of Proposition 6:

Clearly, the left-hand-side of (9) is increasing in $\hat{x}$. First, we establish that the right-hand-side of (9) is decreasing in $\hat{x}$. Then, the equilibrium $\hat{x}$ can be defined as the unique crossing point of those two.

To establish that the right-hand-side of (9) is decreasing in $\hat{x}$, it suffices to show that $s_2(1-\hat{x})$ is decreasing in $\hat{x}$. To see this, rewrite (6) as

$$\alpha\left(1-\frac{1}{2}\frac{u_2\hat{x}}{u_1(1-\hat{x})+u_2\hat{x}}\right)=\delta\frac{m_2}{1-m_2}.$$

From Proposition 4, the right-hand-side is decreasing in $\hat{x}$. Therefore, in the left-hand-side, $u_2\hat{x}/[u_1(1-\hat{x})+u_2\hat{x}]$ has to be increasing in $\hat{x}$. Since

$$s_2(1-\hat{x})=\frac{u_1(1-\hat{x})}{u_1(1-\hat{x})+u_2\hat{x}}=1-\frac{u_2\hat{x}}{u_1(1-\hat{x})+u_2\hat{x}},$$

$s_2(1-\hat{x})$ is decreasing in $\hat{x}$.

Then, the increase in h will shift up the left-hand-side of (9), and increase $\hat{x}$. □

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