Models of Wage Dynamics*

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1 Introduction

This paper introduces four different models of wage dynamics. They are aimed at explaining how wages move over time and what kind of factors affect the dynamics of wages. The models are:

1. Human capital model by Ben-Porath (1967),
2. Job matching model by Jovanovic (1979),
3. Mandatory retirement model by Lazear (1979),
4. Contract model by Harris and Holmstrom (1982).

Each model explains a different aspect of wage dynamics. All the models are simplified from their original form to enhance their exposition.

2 Ben-Porath's Human Capital Investment Model

The first model is the human capital model by Ben-Porath (1967). We develop a simple two-period version. For simplicity, assume:

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- The production of human capital only requires the worker's time.
- To produce future human capital, both current human capital and current investment are used.

2.1 Model

The worker maximizes his lifetime income:

$$\max_{I} \quad w_1 H_1 (1-I) + \frac{1}{1+r} w_2 H_2$$

subject to

$$H_2 = A(IH_1)^{\alpha} + (1-\delta)H_1,$$

where I is the investment level (time devoted for learning activity), w_i is the wage level in period i for each unit of human capital, H_i is the human capital in period i, and r is the interest rate. Second-period human capital, H_2, is the sum of the produced human capital, $A(IH_1)^{\alpha}$, and the undepreciated part of the first-period human capital, $(1-\delta)H_1$. The parameter A represents the worker's *ability to learn.*

Optimal I can be solved as:

$$I = \frac{1}{H_1}\left(\alpha A \frac{1}{1+r}\frac{w_2}{w_1}\right)^{\frac{1}{1-\alpha}}.$$

The optimal investment level is increasing in both the ability to learn, A, and in w_2. It is decreasing in r and w_1, since an increase in either of these increases the opportunity cost of human capital investment. By investing in learning, the agent forgoes the opportunity of earning (and saving) in the first period. The initial human capital level, H_1, has both positive and negative effect. A large H_1 stimulates investment, since H_1 and I are complementary inputs in human capital production. At the same time, a large H_1 implies the opportunity cost for one unit of investment is large. Here, the latter effect dominates.

Earnings in each periods are:

$$w_1 H_1 (1-I) = w_1 \left[H_1 - \left(\alpha A \frac{1}{1+r}\frac{w_2}{w_1}\right)^{\frac{1}{1-\alpha}}\right]$$

and

$$w_2 H_2 = w_2 \left[A\left(\alpha A \frac{1}{1+r}\frac{w_2}{w_1}\right)^{\frac{\alpha}{1-\alpha}} + (1-\delta)H_1\right].$$

The earning growth rate is higher when A and w_2 are higher, and when r and w_1 are lower.

3 Jovanovic's Matching Model

Here, a simple version of Jovanovic (1979) model is constructed. In Jovanovic (1979), the quality of the match and the signal are assumed to follow normal distributions. We assume instead that there are only two possible qualities and two possible signals, so that they each follow a binomial distribution.

Jovanovic's model illustrates the following:

1. Wages rise with tenure.
2. Quitting is negatively correlated with tenure.

3.1 Two-Period Model

The economy consists of workers and firms. There is a continuum of industries, indexed between $[0, 1]$. Each industry has measure zero. Workers live for two periods, and they are endowed with one unit of labor for each period. Production requires only labor, and output is perfectly observable by workers and firms. We assume that each industry is perfectly competitive. The wage contract is formed at the beginning of each period, thus the wage is based on the expected output conditional on the information available at the beginning of the period. Information is assumed to be symmetric between workers and firms. Hence perfect competition ensures that the income of worker i in an industry j at time t, $y_{ij,t}$, equals the expected output conditional on the information available at the beginning of period t, $E[Y_{ij,t}|\Omega_t]$.

Workers are risk neutral while the credit market is perfect, so they only care about the discounted sum of their income, $E[\sum_{t=1}^{2} \beta^{t-1} y_{ij,t}]$. Workers differ in their ability, and the nature of their ability (what kind of jobs they are good at) is unknown to both workers and firms.

For each industry $j \in [0, 1]$ a worker i is in either one of two matches, $m_{ij} = \{\text{fit} = F, \text{unfit} = U\}$. The state of the match, m_{ij}, is unknown to both workers and firms. Prior to participation, the workers and the firm share the belief that

$$P[F] = p,$$

$$P[U] = 1 - p.$$

Output is uncertain: there are two states, $s = \{\text{good} = G, \text{bad} = B\}$. The probability of occurrence of each state is conditional on the match of the worker. This probability is independent across industries. The probability of each state conditional on match is summarized as follows.

$$P[G|F] = \pi_{G|F},$$

$$P[B|F] = \pi_{B|F},$$

$$P[G|U] = \pi_{G|U},$$

$$P[B|U] = \pi_{B|U},$$

where $\pi_{i|j} \in [0,1]$ for $i = \{G, B\}, j = \{F, U\}$. Note that $\pi_{G|F} + \pi_{B|F} = 1$ and $\pi_{G|U} + \pi_{B|U} = 1$. We assume that $\pi_{G|F} > \pi_{G|U}$, which means that it is more likely to get a good result if the worker is fitted to the industry. We assume that all the π's are common across the industries. The output of worker i in industry j, Y_{ij}, is conditional on the state,

$$Y_{ij} = \begin{cases} g & \text{if } s = G \\ b & \text{if } s = B, \end{cases}$$

where $g > b$.

Since the industries and workers are symmetric, we will denote $Y_t \equiv Y_{ij,t}$ and $y_t \equiv y_{ij,t}$. The expected lifetime income of an worker is

$$W = y_1 + \beta E[\max\{y_2^c, y_2^s\}],$$

where y_1 is the earnings in the first period, β is the discount factor, y_2^c is the earnings in the second period if the worker changes the job, and y_2^s is the earning of the second period if he stays in the same industry. The earnings in the second period are conditional on the worker's history. His *history*, $h = \{G, B\}$, is defined by his outcome in the first period.

We can rewrite the first-period income as follows.

$$\begin{aligned} y_1 &= E[Y_1] \\ &= P[G]g + P[B]b \\ &= (P[G,F] + P[G,U])g + (P[B,F] + P[B,U])b \\ &= \{\pi_{G|F} \cdot p + \pi_{G|U} \cdot (1-p)\}g + \{\pi_{B|F} \cdot p + \pi_{B|U} \cdot (1-p)\}b \\ &= (\pi_{G|F} \cdot g + \pi_{B|F} \cdot b)p + (\pi_{G|U} \cdot g + \pi_{B|U} \cdot b)(1-p) \\ &= \bar{Y}_F \cdot p + \bar{Y}_U \cdot (1-p), \end{aligned}$$

where $\bar{Y}_F \equiv E[Y|F] = \pi_{G|F} \cdot g + \pi_{B|F} \cdot b$ and $\bar{Y}_U \equiv E[Y|U] = \pi_{G|U} \cdot g + \pi_{B|U} \cdot b$ are the expected output conditional on the worker is fit ($\bar{Y}_F$) or unfit ($\bar{Y}_U$). Notice that $\bar{Y}_F > \bar{Y}_U$ by $\pi_{G|F} > \pi_{G|U}$ and $g > b$.

The only choice the worker needs to make is whether to switch his job between the first and the second period. The worker will decide whether he changes the job or not based on his history. When he changes his job, he is indifferent among industries, except for the one where he already has had experience. We assume that each industry has measure zero, thus the worker's expected income when he changes his job is the same as his income in the first period. When an worker changes his job, he is starting over again from the same situation as in the first period.

The decision rule of the worker is summarized in the following lemma.

Lemma 1 *The worker remains in his job if the outcome of the first period was good, $h = G$, and changes his job if the outcome is bad, $h = B$.*

Proof.

$h = G$: We get

$$y_2 = E[Y_2|G] = \max\{\bar{Y}_F \cdot p + \bar{Y}_U \cdot (1-p), \bar{Y}_F \cdot P[F|G] + \bar{Y}_U \cdot (1 - P[F|G])\},$$

where the first argument is the expected income when the worker changes His job. The second argument is the expected income when the worker remains in the same job.

By Bayes' rule,

$$P[F|G] = \frac{P[F,G]}{P[G]} = \frac{P[G|F]P[F]}{P[G|F]P[F] + P[G|U]P[U]} = \frac{\pi_{G|F} \cdot p}{\pi_{G|F} \cdot p + \pi_{G|U} \cdot (1-p)}.$$

It is straightforward to see that $P[F|G] > p$ (from the assumption $\pi_{G|F} > \pi_{G|U}$), and the worker always stays in the same job, when $h = G$.

$h = B$: In a similar manner, it is straightforward to see that the worker always changes his job when $h = B$ since $p > P[F|B]$. □

From Lemma 1, we can calculate the (unconditional) expected income of the worker in the second period

$$\begin{aligned} E[y_2] &= E[E[y_2|h]] \\ &= P[G] \cdot E[y_2|G] + P[B] \cdot E[y_2|B] \\ &= \{\pi_{G|F} \cdot p + \pi_{G|U} \cdot (1-p)\} \cdot \frac{\pi_{G|F} \cdot p \cdot \bar{Y}_F + \pi_{G|U} \cdot (1-p) \cdot \bar{Y}_U}{\pi_{G|F} \cdot p + \pi_{G|U} \cdot (1-p)} \\ &\quad + \{\pi_{B|F} \cdot p + \pi_{B|U} \cdot (1-p)\} \cdot \{\bar{Y}_F \cdot p + \bar{Y}_U \cdot (1-p)\} \\ &= \bar{Y}_F \cdot p + \bar{Y}_U \cdot (1-p) + p \cdot (1-p) \cdot \Delta\pi \cdot (\bar{Y}_F - \bar{Y}_U), \end{aligned} \tag{1}$$

where $\Delta\pi \equiv \pi_{G|F} - \pi_{G|U}$. We utilized Lemma 1 in the third equality.

Note that if there is no turnover possibility, $E[y_2] = \bar{Y}_F \cdot p + \bar{Y}_U \cdot (1-p)$. The third term in (1) exhibits the value of the option that he can change the job at the second period. Notice that $p \cdot (1-p)$ is the unconditional variance of y. Variance enters into the expected income since the payoff structure is "convex" (in the sense that the downside risk is avoided) because of the existence of the job turnover possibility. Figure 1 illustrates the case where $\bar{Y}_F = 1$

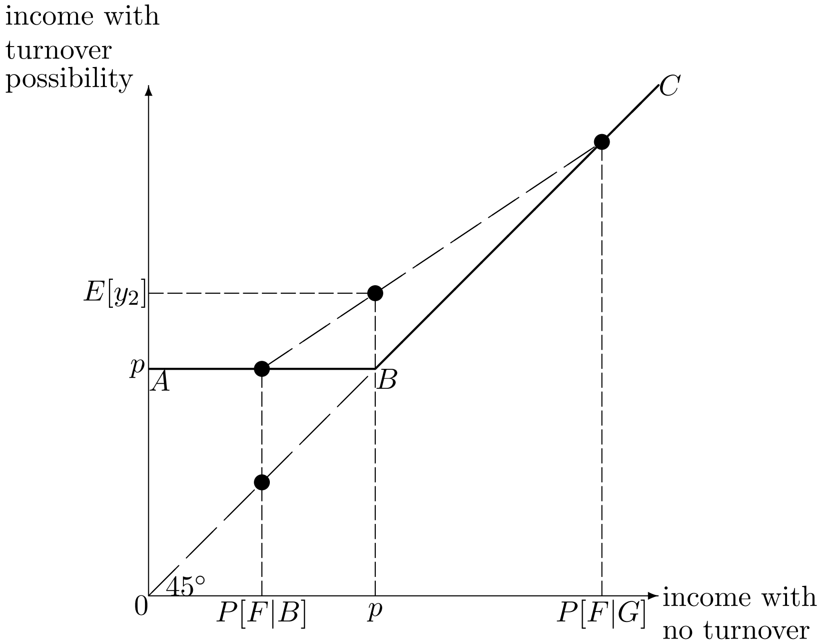

Figure 1:

and $\bar{Y}_U = 0$. When there is no turnover possibility, $E[y_2] = p$. In Figure 1, $E[y_2]$ is larger than p since the payoff function ABC is convex. If the worker observes a bad outcome, he can change his job to get p instead of $P[F|B]$.

The main results are stated in the following proposition.

Proposition 1 *Income rises with tenure, and average income rises with experience.*

Proof.
From the discussion in the proof of Lemma 1, it is straightforward to see that $E[y_1] = E[y_2|B] = \bar{Y}_F \cdot p + \bar{Y}_U \cdot (1-p)$ (people in the first year) and $E[y_2|G] > \bar{Y}_F \cdot p + \bar{Y}_U \cdot (1-p)$ (people in the second year). Average income for workers in the first period is $E[y_1]$, while the average income for workers in the second period is $E[y_2]$, which is larger than $E[y_1]$ from (1). □

Income rises with tenure since only workers who are likely to have a good fit remain in the same job. Average income rises with experience since with longer experience, a worker is more likely to find a job which will be a good fit.

3.2 Three-Period Model

Now, consider a three-period version of the model. Let's start from the third period. In the third period, there are two kinds of workers: workers with one period of tenure (tenure 1) and workers with two periods of tenure (tenure 2). The workers with tenure 1 behave in the same manner as the worker in the two-periods model. Thus, their behavior is dictated by Lemma 1. It turns out that the workers with tenure 2 correspond to the workers who received a good output in period one.[1] It is obvious that they will stay if they receive a good output again in the second period. If they obtain a bad output in the second period, the following holds.

Lemma 2 *When the second period output is bad, the workers will stay in the same job if and only if*

$$\pi_{G|F}(1-\pi_{G|F}) \geq \pi_{G|U}(1-\pi_{G|U}). \tag{2}$$

Proof.
From the Bayes' rule, it is straightforward to see that the conditional probability of fit after experiencing a good then a bad output is

$$P[F|GB] = \frac{\pi_{G|F}(1-\pi_{G|F})p}{\pi_{G|F}(1-\pi_{G|F})p + \pi_{G|U}(1-\pi_{G|U})(1-p)}.$$

It is easy to see that $P[F|GB] \geq p$ if and only if $\pi_{G|F}(1-\pi_{G|F}) \geq \pi_{G|U}(1-\pi_{G|U})$. □

Thus, the decision of a worker with tenure 2 depends on the value of the parameters. Let's analyze two cases in turn.

Case 1: First, consider the case where (2) holds. Then, a worker with tenure 2 never quits. The probability that a worker with tenure 1 quits is

$$p[B] = \pi_{B|F} \cdot p + \pi_{B|U} \cdot (1-p) > 0.$$

Thus, the workers with a longer tenure have a lower tendency to quit.

Case 2: Next, consider the case where (2) does not hold. Then, a worker with tenure 2 will leave if and only if he obtains a bad outcome. The probability of quitting is

$$p[B|y_1 = G] = \pi_{B|F} \cdot P[F|G] + \pi_{B|U} \cdot (1 - P[F|G]).$$

It is clear that $p[B|y_1 = G] < p[B]$ since $P[F|G] > p$. Thus, again, the tendency to quit is lower for tenure 2 workers.

The result is summarized.

Proposition 2 *The worker with tenure 1 is more likely to quit than the worker with tenure 2.*

[1] That is, in the second period, the workers with good period one output stays, while workers with a bad first period output switch. The logic is similar to Lemma 1: with good output, the future perspective is always better by staying rather than switching. With bad output, the future perspective is always better by switching. Formally, this can be seen from the fact that the right-hand side of (1) is increasing in p.

4 Lazear's Model of Mandatory Retirement

This section provides a simple version of Lazear (1979). Lazear's question is: why is there mandatory retirement? By mandatory retirement, he means two (very common) features of a labor contract.

1. There is a definite contract expiration date.

2. At the contract expiration date, workers want to stay at the current wage rate, but the firm is not willing to employ them at this wage. (In other words, [wage]>[marginal product] at the contract expiration date.)

This model also explains that why age-earning profiles are increasing, even if the marginal product is constant.

4.1 Model

Consider the situation where a firm and a worker are trying to write a wage contract. Assume:

- The firm and the worker write a long-term contract (the firm commits to the promised wage as long as they are together), but the firm can lay off the worker when it detects that he "cheated".

- The worker can get a positive benefit from cheating.

- When cheating occurs, the firm can detect it with probability one.

- The firm and the worker only care about the present value of their income streams.

The marginal product of working in the current firm is constant at $MP_t = v$. The marginal product of working outside is increasing with time, $MP_t^o = \tilde{w}_t$. Let $\tilde{w}_0 < v$ and $\tilde{w}_T = v$ at some time T (Figure 2). It is straightforward that, for any contract to be optimal, it has to be terminated at time T.

Property 1 *The contract terminates at time T.*

For simplicity, set the discount rates of the firm and the worker to zero. Thus, the firm cares about the sum of the marginal product minus the wage:

$$\pi = \sum_{t=0}^{T}(v - w_t^*).$$

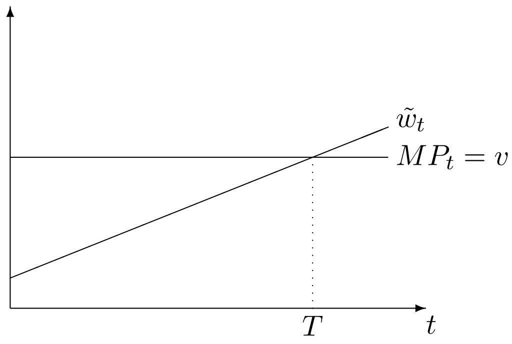

Figure 2:

Assume that, at time zero, there are many such firms. Thus, in equilibrium, $\pi = 0$, and

$$\sum_{t=0}^{T} v = \sum_{t=0}^{T} w_t^* \tag{3}$$

holds.

If the worker does not cheat from time t on, his present-value income from t is:

$$W_t^* \equiv \sum_{s=t}^{T} w_s^* \tag{4}$$

Suppose that if he cheats, he obtains θ_t from cheating. If he cheats, he is laid off, and then he obtains $\tilde{w}_t$ from period t on from the outside option. Thus he compares W_t with

$$Z_t \equiv \sum_{s=t}^{T} \tilde{w}_s + \theta_t$$

and decide whether to cheat. In principle, Z_t can be anything (depending on $\tilde{w}_t$ and θ_t). An example of Z_t and $V_t \equiv \sum_{s=t}^{T} v$ are shown in Figure 3.

To prevent the worker from cheating, the wage schedule has to satisfy

$$W_t^* \geq Z_t \qquad \forall t. \tag{5}$$

In sum, the wage schedule has to satisfy (3) and (5). (3) can be rewritten as

$$W_0^* = V_0.$$

Thus, graphically, W_t^* has to start from A in Figure 3 and always be above the Z_t curve. (It ends at $W_T^* = 0$ if there is no pension.) A typical W_t^* schedule is drawn in Figure 4.

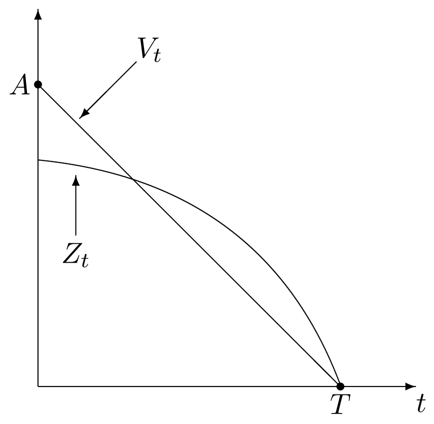

Figure 3:

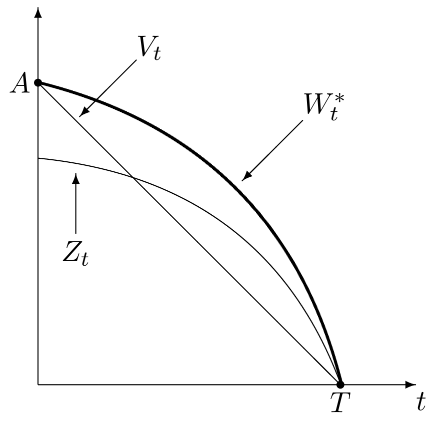

Figure 4:

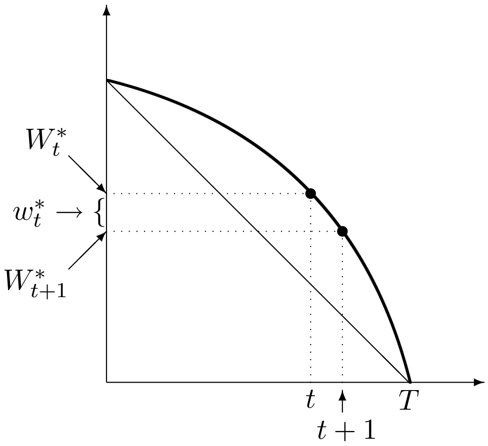

Figure 5:

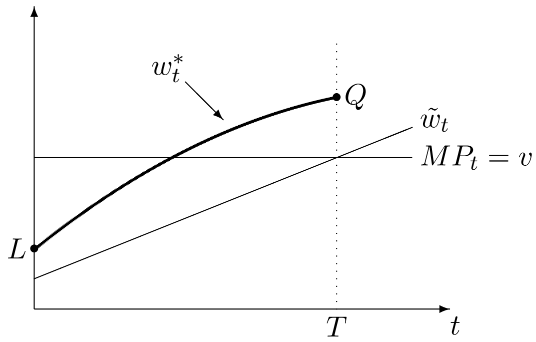

Figure 6:

From (4) notice that

$$w_t^* = W_t^* - W_{t+1}^*,$$

thus the actual wage is the *slope* of W_t^* curve (See Figure 5). It starts from a value lower than v, and ends with a value higher than v. The w_t^* schedule corresponding to W_t^* in Figure 4 is drawn in Figure 6.

Note that the area below LQ is equal to $v \cdot T$ (to satisfy (3)). The wage increases over time even though MP is constant, and especially $w_T^* > v = \tilde{w}_T$. (This comes from the fact that the slope of W_t^* is steeper than the slope of V at time T in Figure 4.) This means that the worker is willing to remain given the wage w_T^*, but the firm doesn't want to keep him at that wage – *mandatory retirement*.

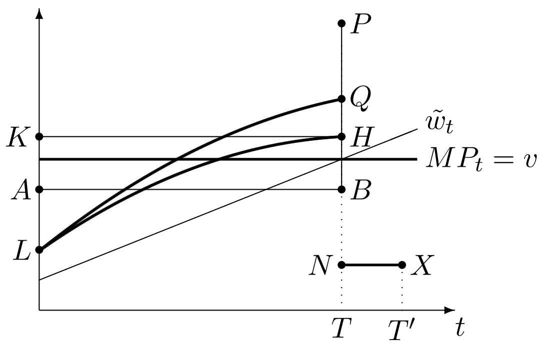

Figure 7:

Property 2 *The retirement is mandatory:*

$$w_T^* > v = \tilde{w}_T.$$

Of course, there are many other possible wage paths that satisfy (3) and (5). Figure 7 is taken from Lazear's original paper. Besides the LQ path, the ABP path (severance payment), the $LKHQ$ path (Becker-Stigler "bonding"), and the $LHNX$ path (pension) are possible.

5 Harris and Holmstrom's Wage Contract Model

This section introduces the wage contract model by Harris and Holmstrom (1982). Here, we prove their Theorem 1 with a simple Lagrangean method. Note that I changed some notation and timing from the original paper.

5.1 Model

Consider a firm-worker match. The production at time t takes place according to:

$$y_t = \eta + \epsilon_t,$$

where y_t is the amount of product produced, η is the worker-specific production ability which is unknown to the firm and the worker, and ϵ_t is random shock which follows i.i.d. $N(0, 1)$. The prior distribution (before observing y_1)[2] of η follows $N(m_1, \sigma_1^2)$. Define the "precision" parameter as $h_t \equiv 1/\sigma_t^2$.

[2]The prior is assumed to be common to the firm and the worker.

Beliefs about ability are updated in Bayesian fashion. Define[3]

$$m_{t+1} = E[\eta | y_t, y_{t-1}, ..., y_1].$$

Then, upon observing y_t, m_{t+1} and h_{t+1} are updated as:

$$m_{t+1} = \frac{h_t m_t + y_t}{h_t + 1},$$

$$h_{t+1} = h_t + 1.$$

Thus, m_t can increase or decrease depending on y_t (but moves less as h increases over time.)

The worker's utility is:

$$\sum_{t=1}^{T} \beta^{t-1} u(w_t),$$

where $u'(\cdot) > 0$ and $u''(\cdot) < 0$: the worker is risk averse. w_t is the wage paid by the firm after observing y_t, and the worker can neither borrow nor save.[4]

The firm's profit is:

$$\pi = \sum_{t=1}^{T} \beta^{t-1} (y_t - w_t),$$

thus the firm is risk neutral. Assume that the labor market is competitive, thus the expected profit of the firm is zero (E_t means the expectation in the beginning of period t, that is, before observing y_t).

$$E_1 \left[\sum_{t=1}^{T} \beta^{t-1} y_t - \sum_{t=1}^{T} \beta^{t-1} w_t \right] = 0. \tag{6}$$

Assume also:

- The firm commits to the contract.
- The worker can walk away from the contract at any point in time ("no slavery").

Therefore, defining $m^t \equiv \{m_1, m_2, ..., m_t\}$ as the information[5] available at time t, the optimal contract solves the history-contingent wage stream[6] $\{w_t(m^t)\}_{t=1}^T$:

$$\max_{\{w_t(m^t)\}_{t=1}^T} \quad E_1 \left[\sum_{t=1}^{T} \beta^{t-1} u(w_t) \right]$$

[3]Note that in the original paper it is defined that $m_{t+1} = E[\eta | y_{t+1}, y_t, y_{t-1}, ..., y_1]$. I changed the notation so that $E[y_{t+1}] = m_{t+1}$ holds, when expectation is taken after observing y_t and before observing y_{t+1}.

[4]The worker wouldn't save under the optimal contract anyway, so only the "no borrowing" assumption is crucial.

[5]Note that, since the output follows a normal distribution (and h_t changes in a deterministic fashion), m^t summarizes all the information available at time t.

[6]Note that the wage is paid before observing y_t. In the original paper, the wage is paid after observing y_t.

subject to

$$E_1\left[\sum_{t=1}^{T}\beta^{t-1}(m_t-w_t)\right]=0, \tag{7}$$

and

$$E_\tau\left[\sum_{t=\tau}^{T}\beta^{t-\tau}(m_t-w_t)\right]\le 0, \qquad \forall\tau. \tag{8}$$

(7) comes from the competitive-firm assumption (6)[7], and (8) comes from the assumption that the worker can leave the contract at any point in time.

To solve this, form a Lagrangean, with the Lagrange multipliers μ for (7) and λ_τ for (8). Then, the FOC for $w_t(m^t)$ is:

$$\beta^{t-1}u'(w_t)-\beta^{t-1}\mu+\sum_{\tau=1}^{t}\beta^{t-\tau}\lambda_\tau=0.$$

Writing the same FOC for w_{t-1}, multiplying β, and subtracting from the above equation, one yields

$$\beta^{t-1}[u'(w_t)-u'(w_{t-1})]+\lambda_t=0.$$

Note that $\lambda_t\ge 0$ since it is a Lagrange multiplier. Thus, $u'(w_t)\le u'(w_{t-1})$, therefore

$$w_t\ge w_{t-1} \tag{9}$$

always holds.

Also, from complementary slackness, $\lambda_t>0$ only if constraint (8) is binding for $\tau=t$. Therefore,

$$\text{If } w_t>w_{t-1},\ \text{then (8) is binding for } \tau=t. \tag{10}$$

In other words, if (8) is not binding, then $w_t=w_{t-1}$. In sum, the wage dynamics are as follows: usually $w_t=w_{t-1}$ holds, but the wage can sometimes jump up to satisfy constraint (8).

Harris and Holmstrom (1982) call the property (9) as "downward wage rigidity" and the property (10) as "upward wage rigidity". They call the wages that satisfy both (9) and (10) as rigid wages. We proved their Theorem 1:

Theorem 1 *The optimal wage policy is a rigid wage policy.*

[7] y_t can be replaced by m_t because $E_1[y_t]=E_1[m_t]$ by the law of iterated expectations.

5.2 A Simpler Model

Here we consider a simpler framework. In particular, consider the two-period Jovanovic model in Section 3, but without the possibility of turnover. Instead of the continuous η in Section 5.1, consider the two states F and U. The expected output in period 1, m_1, is equal to $\bar{Y}_F \cdot p + \bar{Y}_U \cdot (1-p)$. If the worker obtains a good output in the first period, the expected output in the second period $m_{2g} = \bar{Y}_F \cdot P[F|G] + \bar{Y}_U \cdot (1 - P[F|G])$. If the first period output is bad, the second period expected output is $m_{2b} = \bar{Y}_F \cdot P[F|B] + \bar{Y}_U \cdot (1 - P[F|B])$. Clearly $m_{2g} > m_{2b}$ and $p \cdot m_{2g} + (1-p) \cdot m_{2b} = m_1$. Assume that there is no discounting, and workers have a strictly concave utility function $u(\cdot)$. Then, the optimal contract $\{w_1, w_{2g}, w_{2b}\}$ solves

$$\max_{w_1, w_{2g}, w_{2b}} \quad u(w_1) + p \cdot u(w_{2g}) + (1-p) \cdot u(w_{2b})$$

subject to

$$\begin{aligned} w_1 + p \cdot w_{2g} + (1-p) \cdot w_{2b} &= m_1 + p \cdot m_{2g} + (1-p) \cdot m_{2b} \\ &= 2m_1, \end{aligned} \tag{11}$$

$$m_{2g} - w_{2g} \leq 0, \tag{12}$$

$$m_{2b} - w_{2b} \leq 0. \tag{13}$$

Equation (11) corresponds to the zero profit condition (7), and (12) and (13) correspond to the no-slavery condition (8). Let us assign the Lagrange multiplier μ for the constraint (11), and λ_1 and λ_2 for (12) and (13). Then, the Kuhn-Tucker Conditions are:

$$u'(w_1) = \mu, \tag{14}$$

$$u'(w_{2g}) = \mu - \lambda_1 / p, \tag{15}$$

$$u'(w_{2b}) = \mu - \lambda_2 / (1-p), \tag{16}$$

$$\lambda_1 \geq 0, \tag{17}$$

$$\lambda_2 \geq 0, \tag{18}$$

$$\lambda_1 (m_{2g} - w_{2g}) = 0, \tag{19}$$

$$\lambda_2 (m_{2b} - w_{2b}) = 0. \tag{20}$$

There are four cases to consider, from (19) and (20).

1. Case 1: $\lambda_1 = 0$ and $\lambda_2 = 0$.

   From (14)-(16) and (11), $w_1 = w_{2g} = w_{2b} = m_1$. This clearly contradicts to (12), since $m_{2g} > m_1$.

2. Case 2: $\lambda_1 = 0$ and $m_{2b} - w_{2b} = 0$.

   From (14), (15), $m_{2b} = w_{2b}$, and (11), $(1+p)w_{2g} = m_1 + p \cdot m_{2g}$. Again, this contradicts to (12).

3. Case 3: $m_{2g} - w_{2g} = 0$ and $m_{2b} - w_{2b} = 0$.

   From (11), $w_1 = m_1$. This is a contradiction, since (14), (16), and (18) together imply that $w_{2b} \geq w_1$.

4. Case 4: $m_{2g} - w_{2g} = 0$ and $\lambda_2 = 0$.

   This is the only case left. From (14) and (16), $w_{2b} = w_1$ (Downward Rigidity). From (11), it can be solved that $w_1 = w_{2b} = [m_1 + (1-p)m_{2b}]/(2-p)$.

References

[1] Ben-Porath, Y. (1967). "The Production of Human Capital and the Life Cycle of Earnings," *Journal of Political Economy,* 75: 352–365.

[2] Harris, Milton and Bengt Holmstrom (1982). "A Theory of Wage Dynamics," *Review of Economic Studies,* 49: 315–333.

[3] Jovanovic, Boyan (1979). "Job Matching and the Theory of Turnover," *Journal of Political Economy,* 87: 972–990.

[4] Lazear, Edward P. (1979). "Why Is There Mandatory Retirement?" *Journal of Political Economy,* 87: 1261–1284.